BLOCH'S CONJECTURE, DELIGNE COHOMOLOGY AND HIGHER CHOW GROUPS

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Abstract. We express the kernel of Griffiths' Abel-Jacobi map by using the inductive limit of Deligne cohomology in the generalized sense (i.e. the absolute Hodge cohomology of A. Beilinson). This generalizes a result of L. Barbieri-Viale and V. Srinivas in the surface case. We then show that the Abel-Jacobi map for codimension 2 cycles and the Albanese map are bijective if a general hyperplane section is a surface for which Bloch's conjecture is proved. In certain cases we verify Nori's conjecture on the Griffiths group. We also prove a weak Lefschetz-type theorem for (higher) Chow groups, generalize a formula for the Abel-Jacobi map of higher cycles due to Beilinson and Levine to the smooth non proper case, and give a sufficient condition for the nonvanishing of the transcendental part of the image by the Abel-Jacobi map of a higher cycle on an elliptic surface, together with some examples.

Introduction

Let X be a smooth projective complex surface. D. Mumford [41] showed that the kernel of the Albanese map $\mathrm{CH}_0(X)^0 \to \mathrm{Alb}(X)$ is 'huge' if X has a nontrivial global 2-from (i.e. if $p_g(X) \neq 0$). Then S. Bloch [9] conversely conjectured

(0.1) The Albanese map $\mathrm{CH}_0(X)^0 \to \mathrm{Alb}(X)$ is injective if $p_g(X) = 0$.

This conjecture was proved in [13] if X is not of general type, but the general case still remains open, see [3], [31], [54], etc. Related to this, L. Barbieri-Viale and V. Srinivas [2] (see also [29], [46]) constructed an exact sequence

$$H^3_{\mathcal{D}}(X, \mathbb{Z}(2)) \to \varinjlim_U H^3_{\mathcal{D}}(U, \mathbb{Z}(2)) \to \mathrm{CH}_0(X)^0 \to \mathrm{Alb}(X),$$

where U runs over the nonempty open subvarieties of X, and $H^3_{\mathcal{D}}(U, \mathbb{Z}(2))$ denotes Deligne cohomology. This describes the kernel of the Albanese map, and follows also from the local-to-global spectral sequence in the theory of Bloch-Ogus [14].

In this paper, we generalize this to the higher dimensional case, using Deligne cohomology in a generalized sense. The notion of Deligne cohomology was first introduced by

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Deligne in the case X is smooth and proper. It is a natural generalization of the first two terms of the exponential sequence (see [25]). The generalization to the open or singular case was first done by A. Beilinson [4] and H. Gillet [29], where the weight filtration was not used. Later Beilinson [5] found a more natural generalization from the view point of mixed Hodge theory, which he calls absolute Hodge cohomology, and denotes by $H^i_{\mathcal{H}}(X, A(k))$, $H^i_{\mathcal{H}^p}(X, A(k))$, where A is a subring of $\mathbb{R}$. In this paper we denote them by $H^i_{\mathcal{D}}(X, A(k))'$, $H^i_{\mathcal{D}}(X, A(k))''$ respectively, see (1.1).

Let $\mathrm{CH}^p_{\mathrm{hom}}(X)$ be the subgroup of $\mathrm{CH}^p(X)$ consisting of cycles homologically equivalent to zero. There is Griffiths' Abel-Jacobi map to the intermediate Jacobian $J^p(X)$. Its kernel is described by using Deligne cohomology as follows:

0.2. Theorem. *Let X be a smooth proper variety over $\mathbb{C}$. For any integer p, there is a canonical exact sequence*

$$H^{2p-1}_{\mathcal{D}}(X, \mathbb{Z}(p)) \to \varinjlim_{Y} H^{2p-1}_{\mathcal{D}}(X \setminus Y, \mathbb{Z}(p))'' \to \mathrm{CH}^p_{\mathrm{hom}}(X) \to J^p(X),$$

where the inductive limit is taken over the closed subvarieties Y of X with pure codimension $p-1$. Here $H^{2p-1}_{\mathcal{D}}(X \setminus Y, \mathbb{Z}(p))''$ may be replaced by $H^{2p-1}_{\mathcal{D}}(X \setminus Y, \mathbb{Z}(p))'$ in general, and by $H^{2p-1}_{\mathcal{D}}(X \setminus Y, \mathbb{Z}(p))$ if $p = \dim X$.

For the proof of this, we study the cycle map to Deligne homology of a singular variety. Using the compatibility of the cycle map with the localization sequence (2.7), Theorem (0.2) is reduced to

0.3. Theorem. *For a variety Y of pure dimension m, the cycle map induces isomorphisms*

$$cl : \mathrm{CH}^1(Y) \xrightarrow{\sim} H^{\mathcal{D}}_{2m-2}(Y, \mathbb{Z}(m-1))'' = H^{\mathcal{D}}_{2m-2}(Y, \mathbb{Z}(m-1))'.$$

If $m = 1$, this holds also for $H^{\mathcal{D}}_{2m-2}(Y, \mathbb{Z}(m-1))$.

Indeed, using the localization sequence, we can reduce it to the smooth case (see (3.1) below), because a similar isomorphism for $\mathrm{CH}^1(Y, 1)$ is already known, see [32, 3.1]. It is also possible to describe $\mathrm{CH}^1_{\mathrm{hom}}(Y)$ by using the normalization of Y, see (3.5). This may be useful for explicit calculation.

As another application of (0.3), we prove (see (3.13)):

0.4. Theorem. *Let X be a smooth projective complex algebraic variety of dimension n, and Y be an intersection of $n-2$ general hyperplane sections of X. Assume Y has no global 2-forms and the Albanese map is injective for Y, i.e. Bloch's conjecture (0.1) holds. Then the Albanese map for X and the Abel-Jacobi map for cycles of codimension 2 on X are bijective.*

The proof of the injectivity of the Albanese map is relatively easy, and it implies the surjectivity of the Abel-Jacobi map for cycles of codimension 2, see [26]. For the injectivity of the latter we show (see (3.11)):

0.5. Theorem. *Let X be an irreducible smooth proper complex algebraic variety. Let $f : X \to S$ be a surjective morphism to a smooth variety S. Assume that general fibers*

X_s of f are connected and have no global 2-forms, and the Abel-Jacobi map for cycles of codimension 2 on general fibers is injective. If $H^1(X_s,\mathbb{Q}) \neq 0$ for a general $s \in S$, we assume further that $S = \mathbb{P}^1$ and the restriction morphism $H^1(X,\mathbb{Q}) \to H^1(X_s,\mathbb{Q})$ is an isomorphism for a general $s \in S$. Then the kernel of the Abel-Jacobi map tensored with $\mathbb{Q}$ for cycles of codimension 2 on X comes from that on S.

This implies under the assumption of (0.5) that Nori's conjecture [42] on the Griffiths group [30] for X is true if it holds for S (e.g. if $\dim S \leq 2$), see also (3.14) below. Recall that the conjecture predicts an isomorphism between the Griffiths group for cycles of codimension 2 and the quotient of the image of the Abel-Jacobi map divided by the maximal abelian subvariety (and is equivalent to that Abel-Jacobi equivalence is stronger than algebraic equivalence). This conjecture can be deduced from a well-known conjecture of Beilinson [6] and Bloch [9] on a conjectural filtration of the Chow groups (assuming the Hodge conjecture). The hypothesis on the vanishing of $H^1(X_s,\mathbb{Q})$ in (0.5) is satisfied if general fibers X_s are surfaces of general type (see e.g. [52]).

Under the assumption of (0.4), X has no global p-forms for $p > 1$, and this is compatible with [44]. It is conjectured that the injectivity of the Abel-Jacobi map for cycles of codimension 2 on X in (0.4) should hold by assuming only that X has no global 2-form. The hypothesis on Bloch's conjecture (0.1) in (0.4) is satisfied at least if Y is not of general type, see [13] (and also [3], [31], [54], etc.) The assumption on X in (0.4) is satisfied for example by cubic threefolds (see also [17]) and smooth complete intersections of degree $(2,2)$ in $\mathbb{P}^5$. There are other examples because any smooth projective variety is a general hyperplane section of any $\mathbb{P}^1$-bundle having a section over it (choosing a projective embedding appropriately). In some cases we can show that algebraic and homological equivalences coincide for cycles of codimension 2, see [1] and also (3.17) below. Note that the Griffiths group is not finitely generated for general Calabi-Yau threefolds [56].

As another application we have a weak Lefschetz-type theorem for Chow groups (see (3.15), and for a similar assertion about higher cycles, see (3.16)):

0.6. Theorem. *Let X be a smooth projective complex algebraic variety. Take a Lefschetz pencil $f : \widetilde{X} \to S := \mathbb{P}^1$ where $\pi : \widetilde{X} \to X$ is the blow-up along an intersection of two generic hyperplane sections. Let S' be any nonempty open subvariety of S over which f is smooth. Assume $\dim X \geq 4$. Then $\zeta \in \mathrm{CH}^2_{\mathrm{hom}}(X)$ is zero if its restriction to $X_s := f^{-1}(s)$ vanishes for any $s \in S'$.*

Concerning Bloch's conjecture (0.1), it is known [33] that the conjecture is related to the surjectivity of the cycle map

$$cl : \mathrm{CH}^p(X,1) \to H^{2p-1}_{\mathcal{D}}(X,\mathbb{Z}(p))''$$

for certain smooth nonproper varieties X (see also (3.3) below). Here $\mathrm{CH}^p(X,m)$ denotes Bloch's higher Chow group [10]. In the smooth proper case, Beilinson [5] and Levine [37] described the cycle map explicitly by using currents like for Griffiths' Abel-Jacobi map. In this paper, we extend this to an explicit description in the smooth nonproper case, see (4.3).

Let us return to the case of a smooth proper surface X. By [15], the conclusion of (0.1) would imply the decomposability of $\mathrm{CH}^2(X,1)_{\mathbb{Q}}$ (i.e. it is generated by the image

of $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{C}^*$), see also [26]. So it is conjectured that $\mathrm{CH}^2(X,1)_{\mathbb{Q}}$ is decomposable if $p_g(X)=0$. Thus it would be interesting whether the reduced higher Abel-Jacobi map

$$\mathrm{CH}^2_{\mathrm{ind}}(X,1)_{\mathbb{Q}}\to J(H^2(X,\mathbb{Z})(2))_{\mathbb{Q}}/\mathrm{NS}(X)_{\mathbb{Q}}\otimes_{\mathbb{Z}}\mathbb{C}^*, \tag{0.7}$$

which is induced by the above cycle map, is injective in general. Here $\mathrm{CH}^2_{\mathrm{ind}}(X,1)_{\mathbb{Q}}$ is the quotient of $\mathrm{CH}^2(X,1)_{\mathbb{Q}}$ by the image of $\mathrm{Pic}(X)_{\mathbb{Q}}\otimes_{\mathbb{Z}}\mathbb{C}^*$, see (2.2.3) and (4.5.1) below.

This injectivity is related to Voisin's conjecture [55] on the countability of $\mathrm{CH}^2_{\mathrm{ind}}(X,1)_{\mathbb{Q}}$, because the image of the reduced higher Abel-Jacobi map (0.7) is countable [40]. The kernel of this map is isomorphic to

$$\mathrm{Coker}(K_2(\mathbb{C}(X))_{\mathbb{Q}}\to \varinjlim_U \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q},H^2(U,\mathbb{Q})(2)))$$

by [43], [46], where the morphism of $K_2(\mathbb{C}(X))_{\mathbb{Q}}$ is given by $d\log\wedge d\log$ at the level of integral logarithmic forms, and the inductive limit is taken over the nonempty open subvarieties U of X (see also [5], 6.1). This isomorphism follows easily from the localization sequence of mixed Hodge structures together with the fact that the residue of $d\log f\wedge d\log g$ coincides with the logarithmic differential of the tame symbol of $\{f,g\}$ up to sign. It holds also for open subvarieties U if $H^3(U,\mathbb{Q})=0$.

In view of these considerations we are interested in constructing examples of indecomposable higher cycle such that the transcendental part of its image by the higher Abel-Jacobi map does not vanish (i.e. its image is not contained in the image of $F^1H^2(X,\mathbb{C})$ in the Jacobian). We give a sufficient condition for it together with some examples in the case of elliptic surfaces, see (5.2–3) below. (In an earlier version of this paper, such an example was constructed by calculating period integrals of elliptic curves and using double integration.) Another example satisfying the above property is found independently by P. del Angel and S. Müller-Stach [21]. The support of our cycle is irreducible, and is a fiber of an elliptic surface. Such an example does not seem to have appeared in the literature.

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In Sect. 1, we review some basic facts from the theories of Deligne cohomology and mixed Hodge Modules which are needed in this paper. In Sect. 2, we recall the definition of Bloch's higher Chow groups and the cycle map. In Sect. 3 we prove the main theorems (0.2–5) and some related assertions. In Sect. 4 we describe explicitly the cycle map for higher cycles, and construct examples of indecomposable higher cycles in Sect. 5.

In this paper a variety means a separated scheme of finite type over $\mathbb{C}$. All sheaves are considered on the associated analytic spaces, and $H^j(X^{\mathrm{an}},\mathbb{Q})$ is denoted by $H^j(X,\mathbb{Q})$.

1. Deligne cohomology and mixed Hodge Modules

1.1. Deligne cohomology (see [4, 5, 21, 23, 25, 27, 32], etc.) Let X be a smooth variety, and $\overline{X}$ a smooth compactification of X such that $D:=\overline{X}\setminus X$ is a divisor with normal crossings. Let $j:X\to\overline{X}$ denote the inclusion morphism. Let A be $\mathbb{Z}$ or $\mathbb{Q}$ for

simplicity in this paper. Then A-Deligne cohomology $H^i_{\mathcal{D}}(X, A(k))$ is defined to be the i-th hypercohomology group of

$$C^\bullet_{\overline{X}\langle D\rangle}A(k) := C(\mathbf{R}j_*A_X(k) \oplus \sigma_{\geq k}\Omega^\bullet_{\overline{X}}(\log D) \to \mathbf{R}j_*\Omega^\bullet_X)[-1],$$

where $A_X(k) = (2\pi i)^k A_X \subset \mathbb{C}_X$.

For a complex variety X in general, let $K = (K_A, (K_{\mathbb{Q}}, W), (K_{\mathbb{C}}, F, W))$ be the complex of graded-polarizable mixed A-Hodge structures corresponding by [5, 3.11] to the mixed Hodge complex calculating the cohomology of X which is defined by using a simplicial resolution of a compactification of X as in [22]. Let $\mathrm{MHS}(A)^p$ (resp. $\mathrm{MHS}(A)$) denote the abelian category of graded-polarizable (resp. not necessarily graded-polarizable) mixed A-Hodge structures. Then we define A-Deligne cohomology in the generalized sense by

$$\begin{aligned} H^i_{\mathcal{D}}(X, A(k)) &= H^i(C(K_A(k) \oplus F^k K_{\mathbb{C}} \to K_{\mathbb{C}})[-1]), \\ H^i_{\mathcal{D}}(X, A(k))' &= \mathbf{R}\mathrm{Hom}_{\mathrm{MHS}(A)}(A, K(k)[i]), \\ H^i_{\mathcal{D}}(X, A(k))'' &= \mathbf{R}\mathrm{Hom}_{\mathrm{MHS}(A)^p}(A, K(k)[i]). \end{aligned}$$

The last two are called absolute Hodge cohomology in [5], and denoted respectively by $H^i_{\mathcal{H}}(X, A(k))$, $H^i_{\mathcal{H}^p}(X, A(k))$. By loc. cit. we have natural morphisms

$$H^i_{\mathcal{D}}(X, A(k))'' \to H^i_{\mathcal{D}}(X, A(k))' \to H^i_{\mathcal{D}}(X, A(k)).$$

Similarly, let $K' = (K'_A, (K'_{\mathbb{Q}}, W), (K'_{\mathbb{C}}, F, W))$ be the dual of the complex of graded-polarizable mixed A-Hodge structures corresponding by [5, 3.11] to the mixed Hodge complex calculating the cohomology with compact support of X which is defined by using a simplicial resolution of a compactification of X together with that of the divisor at infinity. Then we define A-Deligne homology in the generalized sense by

$$\begin{aligned} H^{\mathcal{D}}_i(X, A(k)) &= H^{-i}(C(K'_A(-k) \oplus F^{-k} K'_{\mathbb{C}} \to K'_{\mathbb{C}})[-1]), \\ H^{\mathcal{D}}_i(X, A(k))' &= \mathbf{R}\mathrm{Hom}_{\mathrm{MHS}(A)}(A, K'(-k)[-i]), \\ H^{\mathcal{D}}_i(X, A(k))'' &= \mathbf{R}\mathrm{Hom}_{\mathrm{MHS}(A)^p}(A, K'(-k)[-i]). \end{aligned}$$

We have also natural morphisms

$$H^{\mathcal{D}}_i(X, A(k))'' \to H^{\mathcal{D}}_i(X, A(k))' \to H^{\mathcal{D}}_i(X, A(k)).$$

If X is smooth of pure dimension n, then $K = K'(n)[2n]$ so that

$$H^{\mathcal{D}}_i(X, A(k)) = H^{2n-i}_{\mathcal{D}}(X, A(n-k)), \tag{1.1.1}$$

and similarly for $H^{\mathcal{D}}_i(X, A(k))'$, $H^{\mathcal{D}}_i(X, A(k))''$.

For a mixed A-Hodge structure $H = (H_A, (H_{\mathbb{Q}}, W), (H_{\mathbb{C}}, F, W))$ and an integer k, we define

$$\begin{aligned} J(H(k)) &= H_{\mathbb{C}}/(H_A(k)_{\mathrm{free}} + F^k H_{\mathbb{C}}), \\ J'(H(k)) &= W_{2k}H_{\mathbb{C}}/((W_{2k}H_A)(k)_{\mathrm{free}} + F^k H_{\mathbb{C}}), \\ J''(H(k)) &= W_{2k-1}H_{\mathbb{C}}/(((W_{2k}H_A)(k)_{\mathrm{free}} + F^k H_{\mathbb{C}}) \cap W_{2k-1}H_{\mathbb{C}}), \end{aligned}$$

where $H_A(k)_{\rm free}=H_A(k)/H_A(k)_{\rm tor}$. We define also

$$F^kW_{2k}H_A(k)=\mathrm{Ker}(H_A(k)\to H_{\mathbb C}/F^kW_{2k}H_{\mathbb C}).$$

(Similarly for $F^kH_A(k)$.) Then we have short exact sequences

$$\begin{gathered}0\to J(H^{i-1}(X,A)(k))\to H^i_{\mathcal D}(X,A(k))\to F^kH^i(X,A)(k)\to 0,\\ 0\to J(H^{i-1}(X,A)(k))'\to H^i_{\mathcal D}(X,A(k))'\to F^kW_{2k}H^i(X,A)(k)\to 0,\\ 0\to J(H^{i-1}(X,A)(k))''\to H^i_{\mathcal D}(X,A(k))''\to F^kW_{2k}H^i(X,A)(k)\to 0,\end{gathered}\tag{1.1.2}$$

because $J'(H(k))=\mathrm{Ext}^1_{\mathrm{MHS}(A)}(A,H(k))$, $J''(H(k))=\mathrm{Ext}^1_{\mathrm{MHS}(A)^p}(A,H(k))$ by [16] and the semisimplicity of polarizable Hodge structures. We have also

$$\begin{aligned}0\to J(H^{\rm BM}_{i+1}(X,A)(-k))''\to H^{\mathcal D}_i(X,A(k))''&\\ \to F^{-k}W_{-2k}H^{\rm BM}_i(X,A)(-k)\to 0,&\end{aligned}\tag{1.1.3}$$

etc. Here $H^{\rm BM}_i(X,A)$ denotes Borel-Moore homology.

It is known that $H^i_{\mathcal D}(X,A(k))$ and $H^{\mathcal D}_i(X,A(k))$ (together with Deligne local cohomology) satisfy the axioms of Bloch-Ogus [14], see [4], [29], [32], etc. In particular, we have a canonical long exact sequence

$$\to H^{\mathcal D}_i(Y,A(k))\to H^{\mathcal D}_i(X,A(k))\to H^{\mathcal D}_i(U,A(k))\to H^{\mathcal D}_{i-1}(Y,A(k))\to\tag{1.1.4}$$

for a closed subvariety Y of X and $U=X\setminus Y$. (This is functorial for Y, U.) Similar assertions hold for $H^{\mathcal D}_i(X,A(k))'$, $H^{\mathcal D}_i(X,A(k))''$. (The assertion for $H^{\mathcal D}_i(X,\mathbb Q(k))''$ follows also from [49] using (1.6) below.) We can similarly define Deligne local cohomology supported on $Y\subset X$. If X is smooth this coincides with Deligne homology of Y.

1.2. Remark. Let $K=(K_{\mathbb Z},(K_{\mathbb Q},W),(K_{\mathbb C};F,W);K'_{\mathbb Q},(K'_{\mathbb C},W))$ be a polarizable mixed Hodge complex in the sense of [5, 3.9]. They are endowed with (filtered) quasi-isomorphisms

$$\begin{gathered}\alpha_1:K_{\mathbb Z}\otimes_{\mathbb Z}\mathbb Q\to K'_{\mathbb Q},\quad \alpha_2:K_{\mathbb Q}\to K'_{\mathbb Q},\\ \alpha_3:(K_{\mathbb Q},W)\otimes_{\mathbb Q}\mathbb C\to(K'_{\mathbb C},W),\quad \alpha_4:(K_{\mathbb C},W)\to(K'_{\mathbb C},W)\end{gathered}$$

such that $(\mathrm{Gr}^W_iK_{\mathbb Q},\mathrm{Gr}^W_i(K_{\mathbb C},F))$ together with the isomorphism in the derived category $\mathrm{Gr}^W_iK_{\mathbb Q}\otimes_{\mathbb Q}\mathbb C=\mathrm{Gr}^W_iK_{\mathbb C}$ induced by α_3,α_4 is a polarizable Hodge complex of weight i in the sense of [22], i.e. $\mathrm{Gr}^W_i(K_{\mathbb C},F)$ is strict and $H^j(\mathrm{Gr}^W_iK_{\mathbb Q},\mathrm{Gr}^W_i(K_{\mathbb C},F))$ is a polarizable Hodge structure of weight $i+j$.

Let $\mathrm{Dec}\,W$ be as in loc. cit. By definition, we have a canonical surjection

$$(\mathrm{Dec}\,W)_0K^j_{\mathbb Q}\to H^j\mathrm{Gr}^W_{-j}K_{\mathbb Q},$$

(and similarly for $K_{\mathbb C},K'_{\mathbb C}$). For a $\mathbb Q$-Hodge structure $H=(H_{\mathbb Q},(H_{\mathbb C},F))$ of weight 0, let

$$H^{(0)}=\mathrm{Hom}_{\mathrm{MHS}}(\mathbb Q,H),$$

which is identified with a subgroup of $H_{\mathbb{Q}}, H_{\mathbb{C}}$. We define the subcomplex $(\mathrm{Dec}\, W)_0^{(0)} K_{\mathbb{Q}}$ of $(\mathrm{Dec}\, W)_0 K_{\mathbb{Q}}$ so that $(\mathrm{Dec}\, W)_0^{(0)} K_{\mathbb{Q}}^j$ is the inverse image of $(H^j \mathrm{Gr}_{-j}^W K)^{(0)}$ by the above morphism (and similarly for $(\mathrm{Dec}\, W)_0^{(0)} K_{\mathbb{C}}$, $(\mathrm{Dec}\, W)_0^{(0)} K'_{\mathbb{C}}$).

Let $(W_0 H^j K_{\mathbb{Q}})^{(0)}$ be the inverse image of $(\mathrm{Gr}_0^W H^j K_{\mathbb{Q}})^{(0)}$ by the projection of $W_0 H^j K_{\mathbb{Q}}$ to $\mathrm{Gr}_0^W H^j K_{\mathbb{Q}}$. Since d_1 of the weight spectral sequence is a morphism of Hodge structures, we can show the canonical quasi-isomorphism

$$\tau_{\le j} (\mathrm{Dec}\, W)_0^{(0)} K_{\mathbb{Q}} / \tau_{<j} (\mathrm{Dec}\, W)_0^{(0)} K_{\mathbb{Q}} \to (W_0 H^j K_{\mathbb{Q}})^{(0)},$$

and similarly for $K_{\mathbb{C}}, K'_{\mathbb{C}}$.

We define a complex $\Gamma(D''_{\mathcal{H}} K)$ to be the single complex associated with

$$K_{\mathbb{Z}} \oplus (\mathrm{Dec}\, W)_0^{(0)} K_{\mathbb{Q}} \oplus F^0 (\mathrm{Dec}\, W)_0^{(0)} K_{\mathbb{C}} \xrightarrow{\phi} K'_{\mathbb{Q}} \oplus (\mathrm{Dec}\, W)_0^{(0)} K'_{\mathbb{C}},$$

where ϕ is induced by $(\alpha_1 - \alpha_2) \oplus (\alpha_3 - \alpha_4)$, and the degree of the source of ϕ is zero. Then, by an argument similar to [5], we can show the isomorphism

$$\mathrm{Hom}_{\mathcal{D}''}(\mathbb{Z}, K) = H^0 \Gamma(D''_{\mathcal{H}} K), \tag{1.2.1}$$

where $\mathcal{D}''$ denotes the category of polarizable mixed Hodge complexes in the sense of [5, 3.9]. So we can define $H^i_{\mathcal{D}}(X, \mathbb{Z}(k))''$ taking a mixed Hodge complex which calculates the cohomology of X as in [22]. Note that (1.2.1) implies the equivalence of categories

$$D^b \mathrm{MHS}(\mathbb{Z})^p \xrightarrow{\sim} \mathcal{D}''$$

in Lemma 3.11 of [5], and that it is easy to show the exact sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}(\mathbb{Z})^p}(\mathbb{Z}, H^{-1} K) \to \mathrm{Hom}_{\mathcal{D}''}(\mathbb{Z}, K) \to \mathrm{Hom}_{\mathrm{MHS}(\mathbb{Z})}(\mathbb{Z}, H^0 K) \to 0,$$

using the truncation τ.

We can similarly define $\Gamma(D'_{\mathcal{H}} K), \Gamma(D_{\mathcal{H}} K)$ to be the single complex associated with

$$K_{\mathbb{Z}} \oplus (\mathrm{Dec}\, W)_0 K_{\mathbb{Q}} \oplus F^0 (\mathrm{Dec}\, W)_0 K_{\mathbb{C}} \to K'_{\mathbb{Q}} \oplus (\mathrm{Dec}\, W)_0 K'_{\mathbb{C}},$$
$$K_{\mathbb{Z}} \oplus K_{\mathbb{Q}} \oplus F^0 K_{\mathbb{C}} \to K'_{\mathbb{Q}} \oplus K'_{\mathbb{C}}$$

respectively. They can be defined also for a mixed Hodge complex K in the sense of [5, 3.2] (where $\mathrm{Dec}\, W$ is replaced by W). Using these, we can also define $H^i_{\mathcal{D}}(X, A(k))'$, $H^i_{\mathcal{D}}(X, A(k))$. Note that $\Gamma(D_{\mathcal{H}} K)$ is canonically isomorphic to

$$K_{\mathbb{Z}} \oplus F^0 K_{\mathbb{C}} \to K'_{\mathbb{C}}, \tag{1.2.2}$$

if there is a canonical morphism $\alpha' : K_{\mathbb{Z}} \to K_{\mathbb{Q}}$ such that $\alpha_1 = \alpha_2 \circ \alpha'$.

1.3. Lemma. *The canonical morphism $H^i_{\mathcal{D}}(X, A(k))' \to H^i_{\mathcal{D}}(X, A(k))$ is an isomorphism if*

$$H^{i-1}(X, A) \text{ and } H^i(X, A) \text{ have weights} \le 2k, \tag{1.3.1}$$

and $H^i_{\mathcal{D}}(X,A(k))'' \to H^i_{\mathcal{D}}(X,A(k))'$ *is an isomorphism if*

$$\mathrm{Gr}^W_{2k}H^{i-1}(X,\mathbb{Q}) \text{ is isomorphic to a direct sum of } \mathbb{Q}(-k). \tag{1.3.2}$$

We have the corresponding assertion for Deligne homology where $H^{i-1}(X,A)$*,* $H^i(X,A)$ *and* k *are replaced respectively by* $H^{\mathrm{BM}}_{i+1}(X,A)$*,* $H^{\mathrm{BM}}_i(X,A)$ *and* $-k$*.*

Proof. This is clear by (1.1.2).

1.4. Remark. Condition (1.3.2) is satisfied for $H^{2p-2}(X\setminus Y,\mathbb{Q})$ with $i=2p-1$ and $k=p$, if X is smooth and Y is a closed subvariety of codimension $\ge p-1$. Indeed, $\mathrm{Gr}^W_{2p}H^{2p-1}_Y(X,\mathbb{Q})$ is a direct sum of $\mathbb{Q}(-p)$. A similar assertion holds also for $\mathrm{Gr}^W_{2-2m}H^{\mathrm{BM}}_{2m-1}(Y,\mathbb{Q})$ if Y is of pure dimension m.

1.5. Mixed Hodge Modules (see [47]). For a variety X we denote by $\mathrm{MHM}(X)$ the abelian category of mixed $\mathbb{Q}$-Hodge Modules on X, and $D^b\mathrm{MHM}(X)$ its derived category consisting of bounded complexes of mixed $\mathbb{Q}$-Hodge Modules. There is a natural functor $\mathrm{rat}: D^b\mathrm{MHM}(X)\to D^b_c(X,\mathbb{Q})$ assigning the underlying $\mathbb{Q}$-complexes where $D^b_c(X,\mathbb{Q})$ denotes the full subcategory of $D^b_c(X^{\mathrm{an}},\mathbb{Q})$ consisting of $\mathbb{Q}$-complexes whose cohomology sheaves are algebraically constructible. We denote by $H^i: D^b\mathrm{MHM}(X)\to\mathrm{MHM}(X)$ the usual cohomology functor.

For morphisms f of algebraic varieties we have canonically defined functors $f_*, f_!, f^*, f^!$ between the derived categories of mixed $\mathbb{Q}$-Hodge Modules. They are compatible with the corresponding functors of $\mathbb{Q}$-complexes via the functor rat. For a closed embedding $i: X\to Y$, the direct image i_* will be omitted sometimes in order to simplify the notation, because

$$i_*: D^b\mathrm{MHM}(X)\to D^b\mathrm{MHM}(Y) \tag{1.5.1}$$

is fully faithful.

If $X=\mathrm{Spec}\,\mathbb{C}$ we have naturally an equivalence of categories

$$\mathrm{MHM}(\mathrm{Spec}\,\mathbb{C}) = \mathrm{MHS}(\mathbb{Q})^p. \tag{1.5.2}$$

Here the right-hand side is as in (1.1). So $\mathrm{MHM}(\mathrm{Spec}\,\mathbb{C})$ will be identified with $\mathrm{MHS}(\mathbb{Q})^p$.

We denote by $\mathbb{Q}(j)$ the mixed Hodge structure of type $(-j,-j)$ whose underlying $\mathbb{Q}$-vector space is $(2\pi i)^j\mathbb{Q}\subset\mathbb{C}$, see [22]. For a variety X with structure morphism $a_X: X\to \mathrm{Spec}\,\mathbb{C}$, we define

$$\mathbb{Q}^H_X(j)=a^*_X\mathbb{Q}(j),\quad \mathbb{D}^H_X(j)=a^!_X\mathbb{Q}(j), \tag{1.5.3}$$

so that $\mathbb{D}^H_X(j)$ is the dual of $\mathbb{Q}^H_X(-j)$. We will write $\mathbb{Q}^H_X$ for $\mathbb{Q}^H_X(0)$, and similarly for $\mathbb{D}^H_X$. If X is smooth of pure dimension n, we have a canonical isomorphism

$$\mathbb{D}^H_X=\mathbb{Q}^H_X(n)[2n]. \tag{1.5.4}$$

1.6. Proposition. *With the notation of* (1.1) *and* (1.5) *we have canonical isomorphisms*

$$H^i_{\mathcal{D}}(X,\mathbb{Q}(k))'' = \mathrm{Ext}^i(\mathbb{Q},(a_X)_*\mathbb{Q}^H_X(k)),$$
$$H_i^{\mathcal{D}}(X,\mathbb{Q}(k))'' = \mathrm{Ext}^{-i}(\mathbb{Q},(a_X)_*\mathbb{D}^H_X(-k)).$$

Proof. In the case $A=\mathbb{Q}$, we have canonical isomorphisms $K=(a_X)_*\mathbb{Q}^H_X$, $K'=(a_X)_*\mathbb{D}^H_X$ by [50].

1.7. Remark. Let X be a reduced variety of pure dimension n, and X_i be the irreducible components of X. Let $\mathrm{Rat}(X)^*=\prod \mathrm{Rat}(X_i)^*$ with $\mathrm{Rat}(X_i)$ the rational function field of X_i. Then by [27, 2.12], [32, 3.1] we have a canonical isomorphism

$$H^{\mathcal{D}}_{2n-1}(X,\mathbb{Z}(n-1)) = \{g\in \mathrm{Rat}(X)^* : \mathrm{div}\, g=0\}, \tag{1.7.1}$$

where $\mathrm{div}\, g=\sum \mathrm{div}\, g_i$ if $g=(g_i)$ with $g_i\in\mathrm{Rat}(X_i)^*$. (Here (1.3.1–2) are satisfied.)

If X is smooth, this is due to [27, 2.12]. In this case, the left-hand side of (1.7.1) is isomorphic to $\mathrm{Ext}^1(\mathbb{Z}_X,\mathbb{Z}_X(1))$ (where Ext^1 is taken in the category of admissible variation of mixed Hodge structures), and the assertion is related with the theory of 1-motives [22], and is more or less well-known. Indeed, if X is a point, the assertion is verified by calculating the period of the mixed Hodge structure on $H^1(\mathbb{A}^1\setminus\{0\},\{1\}\cup\{x\})$ for $x\in\mathbb{C}\setminus\{0,1\}$, i.e., by using the integral of dt/t on the relative cycle connecting $\{1\}$ and $\{x\}$, where t is the coordinate of $\mathbb{A}^1$. The general case is reduced to the smooth case using a long exact sequence, see [32, 3.1].

2. Higher Chow groups and cycle maps

2.1. Higher Chow groups ([10]). Let $\Delta^n=\mathrm{Spec}(\mathbb{C}[t_0,\dots,t_n]/(\sum t_i-1))$. For a subset I of $\{0,\dots,n\}$, let $\Delta^n_I=\{t_i=0\,(i\in I)\}\subset\Delta^n$. It is naturally isomorphic to Δ^m with $m=n-|I|$ (fixing the order of the coordinates), and is called a face of Δ^n. For $0\le i\le n$, we have inclusions $\iota_i:\Delta^{n-1}\to\Delta^n$ such that its image is $\Delta^n_{\{i\}}$.

Let X be an equidimensional variety. Then $X\times\Delta^n_I$ is also called a face of $X\times\Delta^n$. Following Bloch, we define $\mathcal{Z}^p(X,n)$ to be the free abelian group with generators the irreducible closed subvarieties of $X\times\Delta^n$ of codimension p, intersecting all the faces of $X\times\Delta^n$ properly. We have face maps

$$\partial_i:\mathcal{Z}^p(X,n)\to\mathcal{Z}^p(X,n-1),$$

induced by ι_i. Let $\partial=\sum(-1)^i\partial_i$. Then $\partial^2=0$, and $\mathrm{CH}^p(X,n)$ is defined to be $\mathrm{Ker}\,\partial/\mathrm{Im}\,\partial$ which is a subquotient of $\mathcal{Z}^p(X,n)$. By [10] it is isomorphic to

$$\frac{\bigcap_{0\le i\le n}\mathrm{Ker}(\partial_i:\mathcal{Z}^p(X,n)\to\mathcal{Z}^p(X,n-1))}{\partial_{n+1}(\bigcap_{0\le i\le n}\mathrm{Ker}(\partial_i:\mathcal{Z}^p(X,n+1)\to\mathcal{Z}^p(X,n)))} \tag{2.1.1}$$

Indeed, let $\mathcal{Z}^p(X,\bullet)'$ be the subcomplex of $\mathcal{Z}^p(X,\bullet)$ defined by

$$\mathcal{Z}^p(X,n)' = \bigcap\nolimits_{0\le i<n} \mathrm{Ker}(\partial_i : \mathcal{Z}^p(X,n) \to \mathcal{Z}^p(X,n-1)).$$

Then the inclusion induces a quasi-isomorphism

$$\mathcal{Z}^p(X,\bullet)' \to \mathcal{Z}^p(X,\bullet). \tag{2.1.2}$$

(For this, we can consider first the subcomplex defined by $\mathrm{Ker}\,\partial_0$, using a homotopy given by the zeroth degeneracy, and then proceed inductively.)

2.2. Remarks. (i) In this paper we are mainly interested in $\mathrm{CH}^p(X,n)$ for $n=0,1$. If $n=0$, it is the usual Chow group. If $n=1$, any higher cycle $\zeta\in\mathrm{CH}^p(X,1)$ can be represented by $\sum_j(Z_j,g_j)$ where Z_j are irreducible (and reduced) subvarieties of X with pure codimension $p-1$ and g_j are rational functions on Z_j such that $\sum_j \mathrm{div}\, g_j=0$. Indeed, such elements modulo the relation given by the tame symbols form an abelian group $H^{p-1}(X,\mathcal{K}_p)$ using the Gersten resolution, where $\mathcal{K}_p$ is the Zariski-sheafification of the Quillen K-group. It is well known (see e.g. [40]) that there is a natural isomorphism

$$H^{p-1}(X,\mathcal{K}_p) = \mathrm{CH}^p(X,1). \tag{2.2.1}$$

For each $\sum_j(Z_j,g_j)$, the corresponding higher cycle is defined by taking the closure of the graph of g_j in $X\times\mathbb{P}^1$, and then restricting it to the complement of $X\times\{1\}$. Here we use an automorphism of $\mathbb{P}^1$ sending $0,1,\infty$ to $0,\infty,1$ respectively (or rather take another affine coordinate of $\mathbb{P}^1$).

(ii) If we assume $g_j=\mathrm{const}$ in the above Remark, we get a natural morphism

$$\mathrm{CH}^{p-1}(X)\otimes\mathbb{C}^* \to \mathrm{CH}^p(X,1). \tag{2.2.2}$$

Its image is denoted by $\mathrm{CH}^{p-1}_{\mathrm{dec}}(X,1)$, and is called the subgroup of *decomposable* higher cycles, see [19], [40], etc. We define the group of indecomposable higher cycles by

$$\mathrm{CH}^{p-1}_{\mathrm{ind}}(X,1)_{\mathbb{Q}} = \mathrm{CH}^{p-1}(X,1)_{\mathbb{Q}}/\mathrm{CH}^{p-1}_{\mathrm{dec}}(X,1)_{\mathbb{Q}}. \tag{2.2.3}$$

2.3. Functoriality. Let $f:X\to Y$ be a proper morphism of varieties, and put $r=\dim X-\dim Y$. Then we have the pushforward functor

$$f_*:\mathrm{CH}^p(X,n)\to\mathrm{CH}^{p-r}(Y,n).$$

In fact, for a face map $\iota:\Delta^m\to\Delta^n$, Bloch showed the commutative diagram

$$\begin{array}{ccc}
\mathcal{Z}^p(X,n) & \xrightarrow{\iota^*} & \mathcal{Z}^p(X,m) \\
{\scriptstyle f_*}\big\downarrow & & {\scriptstyle f_*}\big\downarrow \\
\mathcal{Z}^{p-r}(X,n) & \xrightarrow{\iota^*} & \mathcal{Z}^{p-r}(X,m)
\end{array}$$

As for the pull-back, we have $f^* : \mathrm{CH}^p(Y,n) \to \mathrm{CH}^p(X,n)$ if f is flat. In the case X, Y are quasi-projective and smooth, we have $f^* : \mathrm{CH}^p(Y,n)_{\mathbb{Q}} \to \mathrm{CH}^p(X,n)_{\mathbb{Q}}$ by [38]. Here we have a quasi-isomorphic subcomplex $\mathcal{Z}^p_f(Y,\bullet)_{\mathbb{Q}}$ of $\mathcal{Z}^p(Y,\bullet)_{\mathbb{Q}}$ on which the pull-back f^* is naturally defined.

2.4. Cycle map. Let X be an equidimensional variety. By [4], [11], [23], etc., we have a cycle map

$$cl : \mathrm{CH}^p(X,n) \to H^{\mathcal{D}}_{2d+n}(X,\mathbb{Q}(d))'', \tag{2.4.1}$$

where $d = \dim X - p$. The target becomes $H^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p))''$ by (1.1.1) if X is smooth. Using mixed Hodge Modules [48], the cycle map (2.4.1) is defined as follows.

Let $S^{n-1} = \bigcup \Delta^n_{\{i\}} \subset \Delta^n, U = \Delta^n \setminus S^{n-1}$ with the inclusion morphisms $i : S^{n-1} \to \Delta^n, j : U \to \Delta^n$. Then

$$(a_{\Delta^n})_* j_! \mathbb{Q}^H_U = \mathbb{Q}^H_{pt}[-n], \tag{2.4.2}$$

where $a_{\Delta^n} : \Delta^n \to pt := \operatorname{Spec} \mathbb{C}$ is the structure morphism. Let $\zeta = \sum_k n_k [Z_k] \in \bigcap_{0 \le i \le n} \operatorname{Ker} \partial_i \subset \mathcal{Z}^p(X,n)$ (see (2.1)), where Z_k are irreducible closed subvarieties of $X \times \Delta^n$. Let $d' = \dim Z_k = d + n$. Put $Z = \bigcup_k Z_k$. Then the coefficients n_k of Z_k induce a morphism

$$\mathbb{Q}^H_Z \to \textstyle\bigoplus_k \mathrm{IC}_{Z_k}\mathbb{Q}^H[-d'] \to \mathbb{D}^H_Z(-d')[-2d'] \to \mathbb{D}^H_{X\times\Delta^n}(-d')[-2d'], \tag{2.4.3}$$

where $\mathrm{IC}_{Z_k}\mathbb{Q}^H$ denotes the mixed Hodge Module whose underlying perverse sheaf is the intersection complex $\mathrm{IC}_{Z_k}\mathbb{Q}$ [7]. Let $\pi : X \times \Delta^n \to X$ be the first projection, and let j denote also $\mathrm{id} \times j : X \times U \to X \times \Delta^n$ (and the same for i). Then

$$\pi_* j_! \mathbb{D}^H_{X\times U} = \mathbb{D}^H_X(n)[n]$$

by (1.5.4) and (2.4.2). So it is enough to show that (2.4.3) is uniquely lifted to

$$\mathbb{Q}^H_Z \to j_! \mathbb{D}^H_{X\times U}(-d')[-2d'],$$

i.e., the composition of (2.4.3) with

$$\mathbb{D}^H_{X\times\Delta^n}(-d')[-2d'] \to i_* i^* \mathbb{D}^H_{X\times\Delta^n}(-d')[-2d']$$

is zero and $\operatorname{Hom}(\mathbb{Q}^H_Z, i_* i^* \mathbb{D}^H_{X\times\Delta^n}(-d')[-2d'-1]) = 0$. But they follow from the condition on proper intersection together with $\zeta \in \bigcap_{0\le i\le n} \operatorname{Ker} \partial_i$. For the well-definedness of the cycle map, it is enough to show its invariance under a deformation of cycle parametrized by $\mathbb{A}^1$ (using a blow-up of Δ^n).

2.5 Remarks. (i) The cycle map for $n = 0$ is defined with integral coefficients by the composition of

$$\mathbb{Z} \to H^{\mathcal{D}}_{2d}(Z,\mathbb{Z}(d)) \to H^{\mathcal{D}}_{2d}(X,\mathbb{Z}(d)) \tag{2.5.1}$$

for $\zeta = \sum_k n_k[Z_k] \in \mathrm{CH}_d(X)$, where $Z = \cup_k Z_k$, see [4], [29], etc. If X is smooth proper, this coincides with Deligne's cycle map, which is defined by the composition of

$$\mathbb{Z} \to H^{2p}_Z(X, \mathbb{Z}(p)) \to K(p)[2p]$$

where K is as in (1.1). It induces Griffiths' Abel-Jacobi map

$$\mathrm{CH}^p_{\mathrm{hom}}(X) \to J^p(X) := \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^{2p-1}(X, \mathbb{Z})(p)), \tag{2.5.2}$$

see [25] (and also [47, (4.5.20)], etc.) Here $J^p(X)$ is Griffiths' intermediate Jacobian by [16]. (This can be defined even if X is not proper.) If $p = \dim X$, then (2.5.2) is the Albanese map.

(ii) If X is purely m-dimensional, the cycle map induces isomorphisms for $n > 0$:

$$\begin{aligned} \mathrm{CH}^1(X, n) &\xrightarrow{\sim} H^{\mathcal{D}}_{2m-2+n}(X, \mathbb{Z}(m-1))'' \\ &= H^{\mathcal{D}}_{2m-2+n}(X, \mathbb{Z}(m-1))' = H^{\mathcal{D}}_{2m-2+n}(X, \mathbb{Z}(m-1)) \end{aligned} \tag{2.5.3}$$

and these groups vanish if $n > 1$. Indeed, let U be a smooth dense open subvariety of X, and Z its complement. Then $\mathrm{CH}^0(Z, n) = 0$ for $n > 0$ by (2.1.1), and $\mathrm{CH}^1(U, n) = 0$ for $n > 1$ by [10, 6.1]. Thus $\mathrm{CH}^1(X, n) = 0$ for $n > 1$ by the localization sequence [12]. On the other hand, the last two isomorphisms of (2.5.3) follow from (1.3), and $H^{\mathcal{D}}_{2m-2+n}(X, \mathbb{Z}(m-1))''$ vanishes for $n > 1$ by (1.1.3). So the case $n > 1$ is clear. The case $n = 1$ is shown by [27, 2.12] in the smooth case and [32, 3.1] (combined with the above (2.2.1)) in general. We will generalize (2.5.3) to the case $n = 0$ in (3.1).

2.6. Compatibility. The cycle map (2.4.1) is compatible with f_* for a proper morphism f, and also with f^* for a morphism of smooth quasi-projective varieties $f : X \to Y$ in the case of rational coefficients. Indeed, this is reduced to the case of the usual Chow groups by (2.3) and the construction of the cycle map (2.4), and follows from [49].

2.7. Proposition. *Let X be a quasi-projective variety, and Y a closed subvariety. Assume X, Y are equidimensional. Let $r = \mathrm{codim}_X Y$, and $d = \dim X - p$. Then the cycle map induces a morphism of long exact sequences*

$$\begin{array}{ccccc} \mathrm{CH}^p(X \setminus Y, n+1) & \longrightarrow & \mathrm{CH}^{p-r}(Y, n) & \longrightarrow & \mathrm{CH}^p(X, n) \\ \downarrow & & \downarrow & & \downarrow \\ H^{\mathcal{D}}_{2d+n+1}(X \setminus Y, \mathbb{Q}(d))'' & \longrightarrow & H^{\mathcal{D}}_{2d+n}(Y, \mathbb{Q}(d))'' & \longrightarrow & H^{\mathcal{D}}_{2d+n}(X, \mathbb{Q}(d))'' \end{array}$$

where the first exact sequence is the localization sequence [12] (*choosing the sign appropriately*), *and the second comes from* (1.1.4).

Proof. The assertion is clear except for the commutativity of the left part of the diagram. Let $U' = \Delta^{n+1} \setminus \cup_{0 \le i \le n} \Delta^{n+1}_{\{i\}}$, and identify $\Delta^{n+1}_{\{n+1\}} \cap U'$ with $U^n := \Delta^n \setminus S^{n-1}$. Let

$j : X \setminus Y \to X, j' : U' \to \Delta^{n+1}, j^n : U^n \to \Delta^n$ denote the inclusion morphisms so that we have distinguished triangles

$$\to \mathbb{D}_Y^H \to \mathbb{D}_X^H \to j_*\mathbb{D}_{X\setminus Y}^H \to,$$
$$\to j_!^{n+1}\mathbb{D}_{U^{n+1}}^H \to j'_!\mathbb{D}_{U'}^H \to j_!^n\mathbb{D}_{U^n}^H(1)[2] \to,$$

where the direct images by closed embeddings are omitted to simplify the notation, see (1.5.1).

Let $\zeta \in \mathrm{CH}^p(X \setminus Y, n+1)$. By (2.1.2) and [12], it is represented by $\zeta \in \mathcal{Z}^p(X \setminus Y, n+1)'$ which is extended to $\zeta' \in \mathcal{Z}^p(X, n+1)'$ so that its restriction to $X \times \Delta^n$ is $\overline{\zeta} \in \mathcal{Z}^p(Y, n)'$. Then the image of ζ by the morphism of the localization sequence is $\overline{\zeta}$. By definition ζ' gives

$$\xi \in \mathrm{Hom}(\mathbb{Q}_Z^H, j'_!\mathbb{D}_{X\times U'}^H(-d'-1)[-2d'-2])$$

such that its restriction to $(X\setminus Y)\times\Delta^n$ vanishes. (Here $d' = \dim X - p + n$, and j' denotes also $id \times j'$.) So it induces

$$\xi' \in \mathrm{Hom}(\mathbb{Q}_Z^H, j_!^n\mathbb{D}_{Y\times U^n}^H(-d')[-2d']),$$
$$\xi'' \in \mathrm{Hom}(\mathbb{Q}_Z^H, j_*j_!^{n+1}\mathbb{D}_{(X\setminus Y)\times U^{n+1}}^H(-d'-1)[-2d'-2]),$$

using the external product of the above two distinguished triangles. We see that ξ' coincides with the image of $\overline{\zeta}$ by the cycle map, and the second distinguished triangle induces an isomorphism $(a_{\Delta^n})_*j_!^n\mathbb{D}_{U^n}^H(1)[2] \to (a_{\Delta^{n+1}})_*j_!^{n+1}\mathbb{D}_{U^{n+1}}^H$. So the assertion is reduced to the next lemma. (Here we represent the middle terms of the distinguished triangles by the mapping cone of the morphism of the other terms so that we get short exact sequences as below.)

2.8. Lemma. *Let $\{K^{i,j,\bullet}\}$ be a square diagram of short exact sequences of complexes of an abelian category, i.e. $K^{i,j,k} = 0$ for $|i| > 1$ or $|j| > 1$, and $K^{i-1,j,k} \to K^{i,j,k} \to K^{i+1,j,k}$ is exact (and similarly for the index j). Assume $H^{k-1}(K^{1,1,\bullet}) = 0$. Let $\xi \in H^k(K^{0,0,\bullet})$ such that its image in $H^k(K^{1,1,\bullet})$ vanishes. Let $\xi' \in H^k(K^{-1,1,\bullet})$, $\xi'' \in H^k(K^{1,-1,\bullet})$ such that the images of ξ, ξ' in $H^k(K^{0,1,\bullet})$ coincide and the images of ξ, ξ'' in $H^k(K^{1,0,\bullet})$ coincide. Then the images of ξ', ξ'' in $H^{k+1}(K^{-1,-1,\bullet})$ coincide up to sign.*

(The proof is straightforward. For a similar assertion, where the triangle is slightly shifted, a proof is given in [34], p. 268.)

3. Proof of main theorems and related assertions

Since Theorem (0.2) follows from (0.3) and (2.7), we first show Theorem (0.3).

3.1. Proof of Theorem (0.3). It is enough to show the assertion for $H_{2m-2}^{\mathcal{D}}(Y, \mathbb{Z}(m-1))''$ by (1.3). We apply (2.7) to Y and a divisor Z on Y containing $\mathrm{Sing}\, Y$. Let $U = Y \setminus Z$. The assertion follows from [49, I, (3.4)] if Y is smooth (i.e. if $U = Y$). Note that we have

the surjectivity of the cycle map $\mathrm{CH}^1(U) \to H^{\mathcal{D}}_{2m-2}(U,\mathbb{Z}(m-1))''$ in loc. cit, because $H^1(U,\mathbb{Z})$ is torsion-free. By [27, 2.12], we have a similar isomorphism

$$\mathrm{CH}^1(U,1) \to H^{\mathcal{D}}_{2m-1}(U,\mathbb{Z}(m-1))''. \tag{3.1.1}$$

So the general case is reduced to the smooth case by using the cycle map of the localization sequence

$$\mathrm{CH}^1(U,1) \to \mathrm{CH}^0(Z) \to \mathrm{CH}^1(Y) \to \mathrm{CH}^1(U) \to 0$$

to the corresponding exact sequence of Deligne homology. Indeed, the cycle maps are compatible with the first morphism $\mathrm{CH}^1(U,1) \to \mathrm{CH}^0(Z)$ (which is given by the divisor map) by using (1.7.1) and [10, 6.1], see [32, 3.1].

3.2. Remark. In general, the isomorphism $\mathrm{CH}^1(Y) = H^{\mathcal{D}}_{2m-2}(Y,\mathbb{Z}(m-1))$ does not hold (even for a smooth Y), see [49, I, (3.5)]. (In Remark (i) of loc. cit. the assumption of the second statement should be replaced by the condition that $H^2(X,\mathbb{Q}) \cap F^1H^2(X,\mathbb{C})$ is not contained in $W_2H^2(X,\mathbb{Q})$.)

3.3. Theorem. *Let X be a connected smooth projective variety. For an open subvariety U of X, let $j_{U,X} : U \to X$ be the inclusion morphism, and $j^*_{U,X}$ the pull-back of Deligne cohomology. Then, for any integer p, the following three conditions are equivalent to each other:*

(*a*) *Griffiths' Abel-Jacobi map $\mathrm{CH}^p_{\mathrm{hom}}(X)_{\mathbb{Q}} \to J^p(X)_{\mathbb{Q}}$ is injective.*

(*b*) $\varinjlim_Y H^{2p-1}_{\mathcal{D}}(X \setminus Y,\mathbb{Q}(p))'/\mathrm{Im}\, j^*_{X\setminus Y,X} = 0$, *where Y runs over the closed subvarieties of X with pure codimension $p-1$.*

(*c*) *The cycle map $\mathrm{CH}^p(X \setminus Y,1)_{\mathbb{Q}} \to H^{2p-1}_{\mathcal{D}}(X \setminus Y,\mathbb{Q}(p))'/\mathrm{Im}\, j^*_{X\setminus Y,X}$ is surjective for any* (*sufficiently large*) *closed subvarieties Y of X with pure codimension $p-1$.*

Proof. The equivalence of (b) and (c) follows from (2.7) together with (2.5.3) for $n = 1$, which we apply to closed subvarieties of pure codimension $p-1$ in X. The equivalence of (a) and (c) follows from (2.7) and (0.3).

3.4. Remarks. (i) We have $H^{2p-1}_{\mathcal{D}}(X \setminus Y,\mathbb{Q}(p))' = H^{2p-1}_{\mathcal{D}}(X \setminus Y,\mathbb{Q}(p))''$ if Y has codimension $\geq p-1$, and $H^{2p-1}_{\mathcal{D}}(X \setminus Y,\mathbb{Q}(p)) = H^{2p-1}_{\mathcal{D}}(X \setminus Y,\mathbb{Q}(p))'$ if furthermore $p = \dim X$. (In this case, condition (a) is related with [44].) The pull-back $j^*_{X\setminus Y,X}$ vanishes if $H^{2p-2}(X,\mathbb{Q})(p-1)$ is generated by algebraic cycle classes and Y is sufficiently large (using (1.1.4)). I am informed that the three conditions (a), (b), (c) are further equivalent to the condition:

(*d*) The cycle map $\mathrm{CH}^p(X \setminus Y,1)_{\mathbb{Q}} \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^{2p-1}(X \setminus Y,\mathbb{Q}(p))$ is surjective for any (sufficiently large) closed subvarieties Y of X with pure codimension p.

This was studied by Jannsen ([33], 9.10), and is equivalent to condition (c) with last $p-1$ replaced by p, see 9.8 in loc. cit.

(ii) The exact sequence of L. Barbieri-Viale and V. Srinivas [2] for cycles of codimension 2 in the introduction follows also from the local-to-global spectral sequence in the theory

of Bloch and Ogus [14] applied to the absolute Hodge cohomology (using (0.3) in the smooth case). Here the flasqueness of $\mathcal{H}^3_{\mathcal{D}}(\mathbb{Z}(2))''$ is clear, because the inductive limit of $H^2_{\mathcal{D}}(U,\mathbb{Z}(1))''$ vanishes where U runs over the nonempty open subvarieties of an irreducible subvariety of codimension 1 in X.

3.5. Relation with the normalization. We can express the subgroup $\mathrm{CH}^1_{\mathrm{hom}}(Y)$ of $\mathrm{CH}^1(Y)$ consisting of Borel-Moore homologically equivalent to zero cycles by using the normalization of Y. This may be useful for explicit calculation.

Let Y be a connected variety of pure dimension m with $Y_k\,(1\le k\le r)$ the irreducible components of Y. Let $\widetilde{Y}$ be the disjoint union of the normalizations $\widetilde{Y}_k$ of Y_k with $\pi:\widetilde{Y}\to Y$ the natural morphism. Let $D=\{y\in Y:|\pi^{-1}(y)|>1\}$. We assume $\widetilde{Y}$ is smooth, D is a smooth closed subvariety of Y with pure codimension one, and $\pi_*\mathbb{Z}_{\widetilde{Y}}|_D$ is a local system. (We may assume these because $\mathrm{CH}^1(Y)$ and $H^{\mathcal{D}}_{2m-2}(Y,\mathbb{Z}(m-1))$ do not change by deleting a closed subvariety of codimension >1.)

Let $\widetilde{D}=\pi^{-1}(D)$, and $\widetilde{D}_i\,(i\in I), D_j\,(j\in J)$ be connected components of $\widetilde{D}, D$. Put

$$I_j=\{i\in I:\widetilde{D}_i\subset\pi^{-1}(D_j)\},\quad I(k)=\{i\in I:\widetilde{D}_i\subset\widetilde{Y}_k\}.$$

We define $\mathcal{E}_j=\mathrm{Ker}(\mathrm{Tr}:\bigoplus_{i\in I_j}\pi_*\mathbb{Z}_{\widetilde{D}_i}\to\mathbb{Z}_{D_j})$, $E_j=H^0(D_j,\mathcal{E}_j)$, and $E=\bigoplus_j E_j$. Let d_i be the degree of $\widetilde{D}_i$ over $\pi(\widetilde{D}_i)$. Then E_j is naturally identified with

$$\{a_i\in\mathbb{Z}\,(i\in I_j):\textstyle\sum_{i\in I_j}d_ia_i=0\}.$$

Let $E'=\bigoplus_{j\in J}H^1(D_j,\mathcal{E}_j)$. (This may have torsion which is related to the cokernel of $\mathrm{CH}^1(\widetilde{Y})\to\mathrm{CH}^1(Y)$.) Then we have an exact sequence

$$\begin{aligned}0\to H^{\mathrm{BM}}_{2m-1}(\widetilde{Y},\mathbb{Z})&\to H^{\mathrm{BM}}_{2m-1}(Y,\mathbb{Z})\to E(m-1)\\ \xrightarrow{\gamma} H^{\mathrm{BM}}_{2m-2}(\widetilde{Y},\mathbb{Z})&\to H^{\mathrm{BM}}_{2m-2}(Y,\mathbb{Z})\to E'(m-1),\end{aligned}\tag{3.5.1}$$

where γ is defined by $(a_i)\to\sum_i a_i cl([\widetilde{D}_i])$. Here $cl([\widetilde{D}_i])$ denotes the cycle class.

We define $E^0=\mathrm{Ker}\,\gamma\subset E$ so that we get

$$0\to H^1(\widetilde{Y},\mathbb{Z})(1)\to H^{\mathrm{BM}}_{2m-1}(Y,\mathbb{Z})(1-m)\to E^0\to 0,\tag{3.5.2}$$

The associated extension class is denoted by $e\in\mathrm{Ext}^1_{\mathrm{MHS}}(E^0,H^1(\widetilde{Y},\mathbb{Z})(1))$. The cycle map induces an isomorphism of exact sequences

$$\begin{array}{ccccccc}E^0&\longrightarrow&\mathrm{CH}^1_{\mathrm{hom}}(\widetilde{Y})&\longrightarrow&\mathrm{CH}^1_{\mathrm{hom}}(Y)&\longrightarrow&0\\ \|&&\|&&\|&&\\ E^0&\longrightarrow&J^1(\widetilde{Y})&\longrightarrow&J^1(Y)^{\mathrm{BM}}&\longrightarrow&0,\end{array}\tag{3.5.3}$$

where $J^1(Y)^{\mathrm{BM}}:=\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z},H^{\mathrm{BM}}_{2m-1}(Y,\mathbb{Z})(1-m))$ and $J^1(\widetilde{Y})$ is as in (2.5.2) (and is a quotient of the Jacobian of a smooth compactification of $\widetilde{Y}$).

Indeed, let $\mathrm{CH}^1(Y)' = \mathrm{Im}(\mathrm{CH}^1(\widetilde{Y}) \to \mathrm{CH}^1(Y))$, $\mathrm{CH}^1_{\mathrm{hom}}(Y)' = \mathrm{CH}^1(Y)' \cap \mathrm{CH}^1_{\mathrm{hom}}(Y)$. Then, for the exactness of the first row, it is sufficient to show

$$\mathrm{CH}^1_{\mathrm{hom}}(Y)' = \mathrm{CH}^1_{\mathrm{hom}}(Y). \tag{3.5.4}$$

This is reduced to the case where the cycle is supported on $\mathrm{Sing}\, Y$, and follows from the localization sequence for Borel-Moore homology. The second row is induced by (3.5.2), and we can show that for $u \in \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, E^0)$,

$$e{\circ}u \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^1(\widetilde{Y}, \mathbb{Z})(1)) \tag{3.5.5}$$

coincides with the image of $\sum_i a_i[\widetilde{D}_i]$ by the Abel-Jacobi map, where $(a_i) = u(1) \in E^0$. This is verified by using a natural morphism of (3.5.2) to

$$0 \to H^1(\widetilde{Y}, \mathbb{Z})(1) \to H^1(\widetilde{Y} \setminus \widetilde{D}, \mathbb{Z})(1) \to H^0(\widetilde{D}, \mathbb{Z}).$$

3.6. Griffiths group. Let $\mathrm{CH}^p_{\mathrm{alg}}(X)$ denote the subgroup of cycles algebraically equivalent to zero, and $\mathrm{CH}^p_{\mathrm{AJ}}(X)$ denote the kernel of Griffiths' Abel-Jacobi map [30]. Let $J^p(X)^{\mathrm{alg}}$ be the image of the Abel-Jacobi map, and $J^p(X)^{\mathrm{ab}}$ the image of $\mathrm{CH}^p_{\mathrm{alg}}(X)$ which is an abelian subvariety of $J^p(X)$. We call $J^p(X)^{\mathrm{ab}}$ and $J^p(X)^{\mathrm{alg}}/J^p(X)^{\mathrm{ab}}$ respectively the abelian and discrete part of the image of the Abel-Jacobi map. Let $\mathrm{Griff}^p(X)$ denote the Griffiths group $\mathrm{CH}^p_{\mathrm{hom}}(X)/\mathrm{CH}^p_{\mathrm{alg}}(X)$, and $\mathrm{Griff}^p_{\mathrm{AJ}}(X)$ the image of $\mathrm{CH}^p_{\mathrm{AJ}}(X)$ in $\mathrm{Griff}^p(X)$. Then we have a commutative diagram of short exact sequences (where the 0 are omitted):

$$\begin{array}{ccccc}
\mathrm{CH}^p_{\mathrm{alg}}(X) \cap \mathrm{CH}^p_{\mathrm{AJ}}(X) & \longrightarrow & \mathrm{CH}^p_{\mathrm{alg}}(X) & \longrightarrow & J^p(X)^{\mathrm{ab}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{CH}^p_{\mathrm{AJ}}(X) & \longrightarrow & \mathrm{CH}^p_{\mathrm{hom}}(X) & \longrightarrow & J^p(X)^{\mathrm{alg}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Griff}^p_{\mathrm{AJ}}(X) & \longrightarrow & \mathrm{Griff}^p(X) & \longrightarrow & J^p(X)^{\mathrm{alg}}/J^p(X)^{\mathrm{ab}}
\end{array}$$

It is known that $\mathrm{CH}^p_{\mathrm{alg}}(X)$ is divisible [13]. By [39], [45] we have

(3.6.1) $\mathrm{CH}^p_{\mathrm{AJ}}(X)$ is torsion-free if $p = 2$ or $\dim X$.

For $p = 2$, this is proved in [39] by using Bloch's cycle map [8]. (See also [53]).

3.7 Remark. For an open subvariety U of X and a positive integer m, consider the short exact sequence

$$0 \to H^2_{\mathcal{D}}(U, \mathbb{Z}(2))/m \to H^2(U, \mathbb{Z}(2)/m) \to {}_m(H^3_{\mathcal{D}}(U, \mathbb{Z}(2))) \to 0, \tag{3.7.1}$$

where $M/m = M/mM$ and ${}_mM = \mathrm{Ker}(m : M \to M)$ for an abelian group M. The first morphism is surjective after taking the inductive limit over U by [39], and the last term

is related to the kernel of the Abel-Jacobi map, see (0.2). So this can be used also for the proof of (3.6.1).

3.8. Proposition. *If $p = 2$, $\mathrm{CH}^2_{\mathrm{alg}}(X) \cap \mathrm{CH}^2_{\mathrm{AJ}}(X)$ is divisible and torsion-free (i.e. a $\mathbb{Q}$-vector space).*

Proof. Since the torsion-freeness follows from (3.6.1), it is enough to show the divisibility. Let $\zeta \in \mathrm{CH}^2_{\mathrm{alg}}(X) \cap \mathrm{CH}^2_{\mathrm{AJ}}(X)$. This comes from $\zeta' \in \mathrm{Pic}^0(Y)$ where Y is a resolution of singularities of a closed subvariety of pure codimension 1 in X. Let $P = \mathrm{Ker}(\mathrm{Pic}^0(Y) \to J^2(X))$. Then P is an extension of a finite group Γ by an abelian variety P^0. Since P^0 is divisible, there is an exact sequence

$$0 \to P^0_{\mathrm{tor}} \to P_{\mathrm{tor}} \to \Gamma \to 0,$$

and it splits. Let $P' = \mathrm{Ker}(\mathrm{Pic}^0(Y) \to \mathrm{CH}^2_{\mathrm{alg}}(X))$. Since P/P' is torsion-free by (3.6.1), we get an isomorphism $P'_{\mathrm{tor}} \to P_{\mathrm{tor}}$, and $P^0 \to P/P'$ is surjective, i.e.

$$P^0/(P^0 \cap P') = P/P'.$$

So the assertion follows, since the left-hand side is divisible.

3.9. Corollary. *If $p = 2$, $\mathrm{Griff}^2_{\mathrm{AJ}}(X)$ is torsion-free.*

Proof. This is clear by (3.8) and (3.6.1), using the left column of the commutative diagram in (3.6).

3.10. Proposition. *Let X be an irreducible smooth proper complex algebraic variety with a surjective morphism $f : X \to S$ whose general fibers X_s are connected and have no global 2-forms. If general fibers have dimension > 2, we assume that the monodromy invariant part of $H^3(X_s, \mathbb{Q})$ vanishes by restricting it to a sufficiently small nonempty open subvariety of X_s for a general $s \in S$. If $\dim S > 1$, we assume further that $H^1(X_s, \mathbb{Q}) = 0$ for a general $s \in S$, and the Abel-Jacobi map is surjective for cycles of codimension 2 on S. Then the Abel-Jacobi map $\mathrm{CH}^2_{\mathrm{hom}}(X) \to J^2(X)$ is surjective and the discrete part $J^2(X)^{\mathrm{alg}}/J^2(X)^{\mathrm{ab}}$ vanishes.*

Proof. Since $J^p(X)^{\mathrm{alg}}/J^p(X)^{\mathrm{ab}}$ is discrete, $J^p(X) = J^p(X)^{\mathrm{alg}}$ if and only if $J^p(X) = J^p(X)^{\mathrm{ab}}$. So the assertion is equivalent to the vanishing of $\mathrm{Gr}^W_3 H^3(U, \mathbb{Q})$ for a sufficiently small non-empty affine open subvariety U. Indeed, let Y be a resolution of singularities of the divisor $X \setminus U$. Then, using the localization sequence, these two conditions are both equivalent to the surjectivity of $H^1(Y, \mathbb{Q}) \to H^3(X, \mathbb{Q})(1)$, or equivalently, of $J^1(Y) \to J^2(X)$.

To show the vanishing of $\mathrm{Gr}^W_3 H^3(U, \mathbb{Q})$, we use the spectral sequence

$$E_2^{p,q} = H^p(S', R^q f_* \mathbb{Q}_U) \Rightarrow H^{p+q}(U, \mathbb{Q}).$$

Here we may assume that S' is a non empty affine open subvariety of S over which f is smooth (shrinking U and S' if necessary) so that the $R^q f_* \mathbb{Q}_U$ are variations of mixed

Hodge structures. The assumption on the Abel-Jacobi map for S is equivalent to the vanishing of $\mathrm{Gr}_3^W H^3(S',\mathbb{Q})$ for S' sufficiently small. So it is enough to show

$$H^{3-q}(S',\mathrm{Gr}_q^W R^q f_*\mathbb{Q}_U)=0 \quad \text{for } 1\le q\le 3,$$

using the spectral sequence associated to the filtration W, because $R^q f_*\mathbb{Q}_U$ has weights $\ge q$ and $H^i(S',\mathrm{Gr}_j^W R^q f_*\mathbb{Q}_U)$ has weights $\ge i+j$.

We have $H^0(S',\mathrm{Gr}_3^W R^3 f_*\mathbb{Q}_U)=0$ by the hypothesis on the invariant part if general fibers have dimension >2, and the surface case follows from the dual of the weak Lefschetz theorem (here we may assume X_s projective). The hypothesis on the 2-forms implies $\mathrm{Gr}_2^W R^2 f_*\mathbb{Q}_U=0$ using the Lefschetz theorem for divisors. Finally $H^2(S',\mathrm{Gr}_1^W R^1 f_*\mathbb{Q}_U)=0$ by the hypothesis in the case $\dim S>1$ (here the curve case is clear because S' is affine). So we get the assertion.

3.11. Proof of Theorem (0.5). By the decomposition theorem [7], there is a noncanonical isomorphism

$$\mathbf{R}f_*\mathbb{Q}_X[\dim X]\simeq \textstyle\bigoplus_{i,Z} K_Z^i[-i], \tag{3.11.1}$$

where Z runs over the irreducible closed subvarieties of S and the K_Z^i are intersection complexes with local system coefficients on Z, see loc. cit. This decomposition holds in the derived category of mixed Hodge Modules [47] on S, and K_Z^i is pure of weight i, and is generically a variation of Hodge structure of weight $i-\dim Z$ (shifted by $\dim Z$). Let U be the open subvariety of S over which f is smooth. Put $n=\dim X-\dim S$, $m=\dim S$. Then $(K_S^i|_U)[-m]$ is the local system $R^{i+n}f_*\mathbb{Q}_X|_U$.

Let $\zeta\in \mathrm{CH}^2_{\mathrm{AJ}}(X)_{\mathbb{Q}}$. We have to show that this comes from S. The restriction of ζ to a general fiber of f is zero by hypothesis. Using the localization sequence and spreading out [9], there is a divisor Σ on S together with $\zeta'\in \mathrm{CH}^1(Z)_{\mathbb{Q}}$ such that $i_*\zeta'=\zeta$ in $\mathrm{CH}^2(X)_{\mathbb{Q}}$ where $Z=f^{-1}(\Sigma)$ with the inclusion $i:Z\to X$. So, by the injectivity of the cycle map in the divisor case (0.3), it is enough to show that its cycle class in Deligne local cohomology $H^4_{\mathcal{D},Z}(X,\mathbb{Q}(2))$ vanishes modifying ζ' by an element of $\mathrm{CH}^1(Z)_{\mathbb{Q}}$ whose image in $\mathrm{CH}^2(X)_{\mathbb{Q}}$ comes from S, because Deligne local cohomology is identified with Deligne homology. Here Deligne cohomology (or homology) means the absolute Hodge cohomology (or homology), and we omit $''$ to simplify the notation. This holds also for the later part of this section.

We define Deligne (local) cohomology with coefficients in mixed Hodge Modules by

$$\begin{aligned} H^{j-i}_{\mathcal{D},\Sigma}(S,K_S^{i-n}(2)[-m]) &= \mathrm{Hom}_{D^b\mathrm{MHM}(S)}(\mathbb{Q}_S, i'_*i'^!K_S^{i-n}(2)[j-i-m]),\\ H^j_{\mathcal{D}}(Z,K_Z^i(2)) &= \mathrm{Hom}_{D^b\mathrm{MHM}(Z)}(\mathbb{Q}_Z, K_Z^i(2)[j]),\end{aligned}$$

where $i':\Sigma\to S$ denotes the inclusion. We may assume that $\Sigma\supset S\setminus U$ replacing Σ if necessary. Then, by (3.11.1), $H^4_{\mathcal{D},Z}(X,\mathbb{Q}(2))$ is isomorphic to

$$(\textstyle\bigoplus_{i\le 2} H^{4-i}_{\mathcal{D},\Sigma}(S,K_S^{i-n}(2)[-m]))\oplus(\bigoplus_{i,Z\ne S} H_{\mathcal{D}}^{4-n-m-i}(Z,K_Z^i(2))). \tag{3.11.2}$$

Here $H^{4-i}_{\mathcal{D},\Sigma}(S,K_S^{i-n}(2)[-m]))=0$ for $i>2$, because

$${}^pH^j i'^! K_S^{i-n}=0 \quad \text{for } j\ne 1, \tag{3.11.3}$$

see [7] for ${}^pH^j$. (Indeed, it vanishes for $j \neq 0,1$ by the localization sequence, because the direct image by an affine open embedding is an exact functor of perverse sheaves. The vanishing for $j=0$ follows from the property of intersection complex that it has no nontrivial subobjects with strictly smaller support.)

Comparing (3.11.2) with a similar decomposition for $H^4_{\mathcal{D}}(X,\mathbb{Q}(2))$, we see that the cycle class of ζ' in $H^4_{\mathcal{D},Z}(X,\mathbb{Q}(2))$ is given by

$$\textstyle\sum_{i\le 2}\xi^i_S \in \bigoplus_{i\le 2} H^{4-i}_{\mathcal{D},\Sigma}(S,K^{i-n}_S(2)[-m]),$$

because $\zeta \in \mathrm{CH}^2_{\mathrm{AJ}}(X)_{\mathbb{Q}}$. Then the assertion follows from (0.3) if $\xi^i_S = 0$ for any i. Since $K^{-n}_S[-m] = \mathbb{Q}_S$, we may assume $\xi^0_S = 0$ modifying ζ' by a cycle coming from S (using (0.3)). Thus it is enough to show $\xi^i_S = 0$ for $i = 1,2$, modifying ζ' by an element of $\mathrm{CH}^1(Z)_{\mathbb{Q}}$ whose image in $\mathrm{CH}^2(X)_{\mathbb{Q}}$ vanishes.

For $i=2$, the variation of Hodge structure $R^2f_*\mathbb{Q}_X|_U$ has level 0, and hence it is associated to an orthogonal representation which has a finite monodromy group. So we may assume that $K^{2-n}_S[-m]$ is a constant sheaf on a smooth variety S, replacing X with a resolution of singularities of the base change of X by a generically finite morphism to S, because the composition of the pull-back and the pushforward of cycles under a generically finite morphism of irreducible proper smooth varieties is the multiplication by the generic degree.

Since $K^{2-n}_S[-m]$ is constant, there are cycles $\Gamma_j \in \mathrm{CH}^1(X)_{\mathbb{Q}}$ such that the stalk of $K^{2-n}_S(1)[-m]$ at $s\in U$ (i.e. $H^2(X_s,\mathbb{Q})(1)$) is generated by the cycle classes of the restrictions of Γ_j to X_s, where $X_s = f^{-1}(s)$. By an argument similar to [49, II] (using the nearby cycle functor), we see that the cycle class of $\Gamma_j|_Z$ in $H^4_{\mathcal{D},Z}(X,\mathbb{Q}(2))$ comes from the cycle class of Γ_j in $H^2_{\mathcal{D}}(X,\mathbb{Q}(1))$ using the composition of

$$H^2_{\mathcal{D}}(X,\mathbb{Q}(1)) \to H^2_{\mathcal{D}}(Z,\mathbb{Q}(1)) \to H^4_{\mathcal{D},Z}(X,\mathbb{Q}(2)), \tag{3.11.4}$$

where the last morphism is induced by the canonical morphism $\mathbb{Q}_Z \to \mathbf{R}\Gamma_Z\mathbb{Q}_X(1)[2]$. Furthermore, using a decomposition similar to (3.11.2) for $H^2_{\mathcal{D}}(X,\mathbb{Q})$, we see that the image of (3.11.4) is contained in the direct sum of $H^{4-i}_{\mathcal{D},\Sigma}(S,K^{i-n}_S(2)[-m])$ with $i\le 2$ under the decomposition (3.11.2).

Since $K^{2-n}_S(2)[-m] = \bigoplus_j \mathbb{Q}_S(1)$ by using Γ_j, we have

$$H^2_{\mathcal{D},\Sigma}(S,K^{2-n}_S(2)[-m]) = \textstyle\bigoplus_j H^2_{\mathcal{D},\Sigma}(S,\mathbb{Q}(1)) = \bigoplus_{j,k} H^2_{\mathcal{D},\Sigma_k}(S,\mathbb{Q}(1)),$$

where the Σ_k are irreducible components of Σ and $H^2_{\mathcal{D},\Sigma_k}(S,\mathbb{Q}(1)) = \mathbb{Q}$. So the morphism to $H^2_{\mathcal{D},\Sigma}(S,K^{2-n}_S(2)[-m])$ is identified with the cycle class map of $[\Sigma_k]$ to $H^2_{\mathcal{D}}(S,\mathbb{Q}(1))$. If the cycle class of $\eta = \sum_k a_k[\Sigma_k]$ vanishes in $H^2_{\mathcal{D}}(S,\mathbb{Q}(1))$, we see that $\Gamma_j\cdot f^*\eta$ is rationally equivalent to zero. Applying this to each factor of ξ^2_S we may assume $\xi^2_S = 0$ by modifying ζ' modulo rational equivalence on X.

For $i=1$, we may assume $K^{1-n}_S \neq 0$, because the assertion is clear otherwise. Then the variation of Hodge structure $K^{1-n}_S[-1]$ is constant on $S\,(=\mathbb{P}^1)$ by hypothesis. Its stalk

is given by $H = H^1(X_s,\mathbb{Q})$ for a general $s \in S$, and is isomorphic to $H^1(X,\mathbb{Q})$. Since $H^2_\Sigma(S,\mathbb{Q}) = \bigoplus_k H^2_{\Sigma_k}(S,\mathbb{Q}) = \bigoplus_k \mathbb{Q}(-1)$, we have

$$\begin{aligned} H^3_{\mathcal{D},\Sigma}(S, K_S^{1-n}(2)[-1]) &= \textstyle\bigoplus_k \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H(1)),\\ H^3_{\mathcal{D}}(S, K_S^{1-n}(2)[-1]) &= \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H(1)), \end{aligned}$$

and $H^3_{\mathcal{D},\Sigma}(S, K_S^{1-n}(2)[-1]) \to H^3_{\mathcal{D}}(S, K_S^{1-n}(2)[-1])$ is identified with the tensor of the degree map $\bigoplus_k \mathbb{Q}[\Sigma_k] \to \mathbb{Q}$ with $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H(1))$ $(= \mathrm{Pic}(X)^0_{\mathbb{Q}})$. So its kernel is generated by cycles rationally equivalent to zero on $S\,(= \mathbb{P}^1)$ tensored with $\mathrm{Pic}(X)^0_{\mathbb{Q}}$, and we may assume $\xi^1_S = 0$ replacing ζ' modulo rational equivalence on X. This completes the proof of (0.5).

3.12. Theorem. *Let X be an irreducible smooth proper complex algebraic variety. Let $f : X \to S$ be a surjective morphism to $S := \mathbb{P}^1$. Assume that general fibers X_s of f are connected and the restriction morphism $H^1(X,\mathbb{Q}) \to H^1(X_s,\mathbb{Q})$ is an isomorphism for a general $s \in S$. Then the Albanese map for X is injective if it is injective for general fibers X_s.*

Proof. The argument is similar to (3.11). It is enough to consider the Albanese map tensored with $\mathbb{Q}$. Let $\zeta \in \mathrm{CH}^n_{\mathrm{AJ}}(X)_{\mathbb{Q}}$ with $n = \dim X$. We may assume that $f(\mathrm{supp}\,\zeta)$ is contained in the open subvariety U over which f is smooth. Let S' be a multivalued section of f (i.e. finite over S). Let d be its degree over S. Then $f^*f_*\zeta$ and $[S']\cdot f^*f_*\zeta$ are rationally equivalent to zero, and we may assume that $f_*\zeta = 0$ without any equivalence relation, modifying ζ by $d^{-1}[S']\cdot f^*f_*\zeta$.

Now ζ determines an element ξ_s of $\mathrm{Alb}(X_s)\otimes_{\mathbb{Z}}\mathbb{Q}$ for each $s \in f(\mathrm{supp}\,\zeta)$. By hypothesis the Albanese varieties $\mathrm{Alb}(X_s)$ form a constant abelian scheme over U (using duality), and any section is constant because $S = \mathbb{P}^1$. Furthermore, its generic fiber is the Albanese variety of the generic fiber X_K of f (where $K = \mathbb{C}(S)$), and the Albanese map for X_K tensored with $\mathbb{Q}$ is surjective (using the base changes by finite extensions of K). So we can apply an argument similar to the case $i = 1$ in (3.11), and we may assume that $\xi_s = 0$ modifying ζ. Then $\zeta = 0$ by hypothesis, and the assertion follows.

3.13. Proof of Theorem (0.4). The injectivity of the maps follows by applying (0.5), (3.12) inductively to a Lefschetz pencil. The surjectivity of the Albanese map is clear. For cycles of codimension 2 it follows from (3.10) or [26] (using the injectivity).

3.14. Theorem. *Let X be an irreducible smooth proper complex algebraic variety with a surjective morphism $f : X \to S$ to a smooth curve S. Assume that general fibers of f are connected and have no global 2-forms, and the Abel-Jacobi map for cycles of codimension 2 on general fibers is injective. Then Nori's conjecture* [42] *holds, i.e.* $\mathrm{CH}^2_{\mathrm{AJ}}(X) \subset \mathrm{CH}^2_{\mathrm{alg}}(X)$.

Proof. The argument is similar to (3.11) except that we consider the cycle class in the usual cohomology instead of Deligne cohomology, and rational equivalence is replaced by algebraic equivalence. For $i = 1$, we also use the vanishing of $H^j_\Sigma(S, K_S^{1-n}[-1])$ for $j \ne 2$, which follows from (3.11.3) because $\dim\Sigma = 0$. Then the assertion follows by an argument similar to (3.11).

3.15. Proof of Theorem (0.6). The argument is similar to (3.11). By the weak Lefschetz theorem, ζ belongs to $\mathrm{CH}^2_{\mathrm{AJ}}(X)$. Since it is enough to show $\pi^*\zeta = 0$ in $\mathrm{CH}^2(\widetilde{X})$, we may replace X with $\widetilde{X}$. So $\widetilde{X}$ will be denoted by X, and $\pi^*\zeta$ by ζ from now on.

By spreading out there exist Σ and ζ' as in (3.11) such that $i'_*\zeta' = \zeta$ in the notation of (3.11). Since $\dim X_s \ge 3$, the monodromy of $R^2 f_*\mathbb{Q}_X$ is trivial, and $K_S^{2-n}[-1]$ is a constant variation of Hodge structure. Hence the restriction morphism $H^2(X,\mathbb{Q}) \to H^2(X_s,\mathbb{Q})$ is surjective and the Picard number of X_s is independent of $s \in U$. So the assertion follows by the same argument as in (3.11).

3.16. Theorem. *With the notation of* (0.6), *assume* $\dim X \ge 3$. *Then* $\zeta \in \mathrm{CH}^2(X,1)_{\mathbb{Q}}$ *is zero if its restriction to* X_s *vanishes for any* $s \in S'$. *In particular, the higher Abel-Jacobi map*

$$cl : \mathrm{CH}^2(X,1)_{\mathbb{Q}} \to J(H^2(X,\mathbb{Q})(2)) \tag{3.16.1}$$

is injective if this map is injective for any X_s $(s \in S')$.

Proof. The argument is similar to (3.11) and (3.15). It is sufficient to show the first assertion. Since $R^1 f_*\mathbb{Q}_X$ is a constant local system by hypothesis, it does not contribute to the Deligne local cohomology by a weight argument. Then, using the decomposition (3.11.1), the assertion is reduced to the isomorphism between $\mathrm{CH}^1(Z,1)$ and the corresponding Deligne homology [32, 3.1].

3.17. Remarks. (i) The injectivity of the higher Abel-Jacobi map (3.16.1) implies $H^1(X,\mathbb{Q}) = 0$, see [51, 5.2]. It is conjectured that the converse is also true.

(ii) If the assumptions of (3.10) are further satisfied in (3.14) (e.g. if $\dim X = 3$), then algebraic and homological equivalences coincide for cycles of codimension 2 on X by (3.10) and (3.14). (It is conjectured that this should hold assuming only the nonexistence of global 3-forms.) In the case $\dim X = 3$ and $\dim S = 1$, we can prove the assertion also by using Cor. 2.3 in [1]. Indeed, let U be an open subvariety over which f is smooth, and put $Y = f^{-1}(U)$. Then by hypothesis and spreading out ([9], [15]), the diagonal cycle $[Y]$ in $Y \times_U Y$ is rationally equivalent to a cycle $\Gamma_1 + \Gamma_2$ (with rational coefficients) such that the j-th projection of Γ_j to Y is contained in a divisor for $j = 1, 2$ (shrinking U if necessary). Using the embeddings $Y \times_U Y \to Y \times Y \to X \times X$, a similar assertion holds for the diagonal X of $X \times X$, and we can apply the theory of Barbieri-Viale on balanced varieties in loc. cit. The key point is that $\mathrm{Griff}^2(X)$ is a quotient of $H^0_{\mathrm{Zar}}(X, \mathcal{H}^3(\mathbb{Z}(2)))$ by [14] using the local-to-global spectral sequence, and we have the action of the diagonal on $H^0_{\mathrm{Zar}}(X, \mathcal{H}^3(\mathbb{Z}(2)))$ which vanishes up to torsion.

(iii) Nori's conjecture stated in the introduction is equivalent to the one in [42] modulo Grothendieck's generalized Hodge conjecture. Indeed, the last conjecture implies that the abelian part of the image of the Abel-Jacobi map coincides with the abelian variety corresponding to the largest Hodge structure of level ≤ 1 contained in $H^3(X,\mathbb{Q})$.

(iv) It is not easy to generalize the argument in (3.11) to the case $p_g(X_s) \ne 0$ even if we assume $\Gamma(X, \Omega^3_X) = 0$, because the existence of the transcendental part of $H^2(X_s,\mathbb{Q})$ makes the situation completely different (e.g. the Picard number of X_s is not constant).

(v) By a well-known conjecture of Beilinson [6] and Bloch [9], $\mathrm{CH}^2_{\mathrm{AJ}}(X)_{\mathbb{Q}}$ should be determined by $H^2(X,\mathbb{Q})$ (more precisely, it should be expressed by $\mathrm{Ext}^2(\mathbb{Q},H^2(X,\mathbb{Q})(2))$ in the conjectural category of mixed motives). Let Y be a general surface in X which is an intersection of general hyperplane sections. If the Hodge conjecture is true, there is a cycle $\Gamma\in\mathrm{CH}^2(Y\times X)_{\mathbb{Q}}$ such that the composition of the restriction morphism $H^2(X,\mathbb{Q})\to H^2(Y,\mathbb{Q})$ with $\Gamma_*: H^2(Y,\mathbb{Q})\to H^2(X,\mathbb{Q})$ is the identity on $H^2(X,\mathbb{Q})$, and Nori's conjecture can be reduced to the surface case (where the conjecture is trivial) if the conjecture of Beilinson and Bloch is true. Note that the last conjecture can be replaced by their conjecture on the injectivity of the Abel-Jacobi map for smooth projective varieties over number fields, see [51, 0.4].

(vi) Let $\pi: X\to Y$ be a $\mathbb{P}^1$-bundle over a smooth projective variety Y, which has a section D. If D' is a sufficiently very ample divisor on Y, then $D+\pi^*D'$ is very ample on X. Let Z be a general hyperplane section of it. Then Z is a section of π by calculating the intersection with $\pi^{-1}(y)$; in particular, $Y=Z$.

4. Higher Abel-Jacobi map

4.1. Currents. For a complex manifold X of dimension n, let $\mathcal{C}^\bullet(X)$ denote the complex of currents on X which has the Hodge filtration F as usual. Here we normalize $\mathcal{C}^\bullet(X)$ so that $\mathcal{C}^j(X)=0$ for $i>0$ or $i<-2n$. It has a structure of double complex such that the Hodge filtration is given by the first degree. We have a natural morphism $\mathcal{E}^\bullet(X)(n)[2n]\to\mathcal{C}^\bullet(X)$, where $\mathcal{E}^i(X)$ denotes the vector space of C^∞ i-forms on X. Let $S^\bullet(X)$ denote the complex of locally finite C^∞ chains on X where $S^{-j}(X)$ consists of locally finite j-chains so that $H^j(S^\bullet(X))=H^{j+2n}(X,\mathbb{Z})(n)$. There is a natural morphism $\iota: S^\bullet(X)\to\mathcal{C}^\bullet(X)$. The differential d of $S^\bullet(X)$ is defined in a compatible way with that of $\mathcal{C}^\bullet(X)$. So it differs from the usual boundary map ∂ by the sign $-(-1)^{\deg}$ due to the Stokes theorem. Note that the differential of a current Φ is defined by $(d\Phi)(\omega)+(-1)^{\deg\Phi}\Phi(d\omega)=0$ for C^∞ forms ω with compact supports. For a smooth complex algebraic variety X, we will denote $S^\bullet(X^{\mathrm{an}})$, $\mathcal{C}^\bullet(X^{\mathrm{an}})$ by $S^\bullet(X)$, $\mathcal{C}^\bullet(X)$ to simplify the notation.

Let $\overline{X}$ be a smooth proper complex algebraic variety of dimension n, and D a divisor on $\overline{X}$ with normal crossings such that each irreducible component D_j is smooth. Put $X=\overline{X}\setminus D$. Let $\widetilde{D}^{(j)}$ be the disjoint union of the intersections of j irreducible components as in [22, II]. Then we have naturally a double complex

$$\to F^k\mathcal{C}^\bullet(\widetilde{D}^{(j+1)})\to F^k\mathcal{C}^\bullet(\widetilde{D}^{(j)})\to\cdots\to F^k\mathcal{C}^\bullet(\widetilde{D}^{(0)})\to 0$$

by the dual construction of [22, III] (using the push-down of currents instead of the pull-back of forms), and the associated single complex will be denoted by $F^k\mathcal{C}^\bullet(\overline{X}\langle D\rangle)$.

We have the weight filtration W on $\mathcal{C}^\bullet(\overline{X}\langle D\rangle)$ such that $\mathrm{Gr}^W_j\mathcal{C}^\bullet(\overline{X}\langle D\rangle)=\mathcal{C}^\bullet(\widetilde{D}^{(j)})[j]$. We define similarly $S^\bullet(\overline{X}\langle D\rangle)$ with the filtration W such that $\mathrm{Gr}^W_jS^\bullet(\overline{X}\langle D\rangle)=S^\bullet(\widetilde{D}^{(j)})[j]$. Then we get the polarizable mixed Hodge complex $K(\overline{X}\langle D\rangle)$ defined by

$$(S^\bullet(X),(S^\bullet(\overline{X}\langle D\rangle)_{\mathbb{Q}},W),(\mathcal{C}^\bullet(\overline{X}\langle D\rangle);F,W);S^\bullet(X)_{\mathbb{Q}},(\mathcal{C}^\bullet(\overline{X}\langle D\rangle),W)),$$

which calculates the Borel-Moore homology of X. By (1.2.1) we have a canonical isomorphism (see also [32], [35]):

$$H^i_{\mathcal{D}}(X,\mathbb{Z}(k)) = H^{i-2n}(\Gamma(D''_{\mathcal{H}}(K(\overline{X}\langle D\rangle)(k-n)))). \tag{4.1.1}$$

4.2. Cycle class. With the above notation, let $\zeta = \sum_j (Z_j, g_j) \in \mathrm{CH}^p(X,1)$ as in (2.2). Put $d = n-p$. Let γ_j be the closure of the inverse image by g_j of

$$\{z \in \mathbb{C} \,|\, \mathrm{Re}\, z > 0, \mathrm{Im}\, z = 0\} \subset \mathbb{P}^1.$$

Using a triangulation, it is viewed as a topological chain. We give it an orientation so that $\partial\gamma_j = \mathrm{div}\, g_j$. Then $\gamma := \sum_j \gamma_j$ is a topological cycle on $Z := \cup_j Z_j$, and it belongs to $S^{-2d-1}(X)$. Let $\widetilde{Z}_j \to Z_j$ be a resolution of singularities such that the divisor of the pull-back $\widetilde{g}_j$ of g_j to $\widetilde{Z}_j$ has normal crossings. Let $\pi_j : \widetilde{Z}_j \to X$ denote its composition with the inclusion $i_j : Z_j \to X$. Then we have the push-down of currents $(\pi_j)_* : \mathcal{C}^\bullet(\widetilde{Z}_j) \to \mathcal{C}^\bullet(X)$.

Let $\log_{\mathrm{Hv}} \widetilde{g}_j$ denote a locally integrable function on $\widetilde{Z}_j$ which is defined by choosing a branch of $\log \widetilde{g}_j$ on $\widetilde{Z}_j \setminus \widetilde{\gamma}_j$ where $\widetilde{\gamma}_j$ is the pull-back of γ_j to $\widetilde{Z}_j$. (Hv stands for Heaviside.) Then it is a current on $\widetilde{Z}_j$, and we can verify

$$\begin{aligned} d(\log_{\mathrm{Hv}} \widetilde{g}_j) &= \widetilde{g}_j^{-1} d\widetilde{g}_j - (2\pi i)\iota\widetilde{\gamma}_j, \\ d(\widetilde{g}_j^{-1} d\widetilde{g}_j) &= (2\pi i)\iota(\mathrm{div}\, \widetilde{g}_j). \end{aligned} \tag{4.2.1}$$

For example, we get the first equality by using the integration on the inverse image of

$$\{z \in \mathbb{C} \,|\, \min\{|\arg z|, |z|, |z|^{-1}\} > \varepsilon\}$$

for $\varepsilon \to 0$. Note that $\widetilde{g}_j^{-1} d\widetilde{g}_j$ is a form with locally integrable coefficients on $\widetilde{Z}_j$ and $\sum_j (\pi_j)_*(\widetilde{g}_j^{-1} d\widetilde{g}_j)$ is a closed current. We define

$$\begin{aligned} (i_j)_*(g_j^{-1} dg_j) &= (\pi_j)_*(\widetilde{g}_j^{-1} d\widetilde{g}_j), \\ (i_j)_*(\log_{\mathrm{Hv}} g_j) &= (\pi_j)_*(\log_{\mathrm{Hv}} \widetilde{g}_j), \\ gdg &= \textstyle\sum_j (i_j)_*(g_j^{-1} dg_j) \in F^{-d}\mathcal{C}^{-2d-1}(\overline{X}), \\ \log_{\mathrm{Hv}} g &= \textstyle\sum_j (i_j)_*(\log_{\mathrm{Hv}} g_j) \in \mathcal{C}^{-2d-2}(\overline{X}). \end{aligned}$$

These are independent of the choice of $\widetilde{Z}_j$.

Let $\overline{Z}_j$ be the closure of Z_j in $\overline{X}$. Then g_j is identified with a rational function $\overline{g}_j$ on $\overline{Z}_j$, and we can define $\overline{g}^{-1} d\overline{g} = \sum_j (\overline{i}_j)_*(\overline{g}_j^{-1} d\overline{g}_j)$, etc. similarly, where $\overline{i}_j : \overline{Z}_j \to \overline{X}$ is the inclusion morphism.

Let $\mathrm{div}\, \overline{g} = \sum_j \mathrm{div}\, \overline{g}_j$. This is supported on D, and there is a cycle $(\mathrm{div}\, \overline{g})^{(1)}$ on $\widetilde{D}^{(1)}$ such that its image in $\overline{X}$ coincides with $\mathrm{div}\, \overline{g}$. Taking a triangulation, $(\mathrm{div}\, \overline{g})^{(1)}$ can be viewed as an element of $S^{-2d}(\widetilde{D}^{(1)})$. So we get

$$(g^{-1}dg)^\wedge := (\overline{g}^{-1} d\overline{g}, -(2\pi i)\iota(\mathrm{div}\, \overline{g})^{(1)}) \in F^{-d}\mathcal{C}^{-2d-1}(\overline{X}\langle D\rangle)$$

such that $d(g^{-1}dg)^\wedge=0$. Let $\overline{\gamma}$ be the closure of γ in $\overline{X}$. Since $\partial\overline{\gamma}=\operatorname{div}\overline{g}$, we get

$$\gamma^\wedge := (\overline{\gamma}, -(\operatorname{div}\overline{g})^{(1)}) \in S^{-2d-1}(\overline{X}\langle D\rangle)$$

such that $d\iota\gamma^\wedge=0$. Then $d(\log_{\mathrm{Hv}} g)\in\mathcal{C}^{-2d-1}(\overline{X}\langle D\rangle)$ coincides with the sum of $-(2\pi i)\iota\gamma^\wedge$ and $(g^{-1}dg)^\wedge$ in $\mathcal{C}^{-2d-1}(\overline{X}\langle D\rangle)$ by (4.2.1).

4.3. Theorem. *With the above notation, $cl(\zeta)\in H^{2p-1}_{\mathcal{D}}(X,\mathbb{Z}(p))''$ corresponds by the isomorphism* (4.1.1) *to*

$$(-(2\pi i)^{-d}\gamma, -(2\pi i)^{-d}\gamma^\wedge, -(2\pi i)^{-d-1}(g^{-1}dg)^\wedge; 0, (2\pi i)^{-d-1}\log_{\mathrm{Hv}} g) \tag{4.3.1}$$

where these elements belong to $S^{-2d-1}(X)(-d)$, $S^{-2d-1}(\overline{X}\langle D\rangle)_{\mathbb{Q}}(-d)$, $F^{-d}\mathcal{C}^{-2d-1}(\overline{X}\langle D\rangle)$, $S^{-2d-2}(X)_{\mathbb{Q}}(-d)$ and $\mathcal{C}^{-2d-2}(\overline{X}\langle D\rangle)$ respectively.

Proof. Since the class of $(\operatorname{div}\overline{g})^{(1)}$ in $H^{2p-2}(\widetilde{D}^{(1)},\mathbb{Q})(p-1)$ is a Hodge cycle, we see that (4.3.1) belongs to $H^{-2d-1}(\Gamma(D''_{\mathcal{H}}(K(\overline{X}\langle D\rangle)(-d)))$, see Remark (1.2). Let $\overline{Z}$ be the closure of Z in $\overline{X}$, and

$$\Sigma = \cup_j \operatorname{supp}\operatorname{div}\overline{g}_j \cup \operatorname{Sing}\overline{Z}.$$

Then the canonical morphism

$$H^{2p-1}_{\mathcal{D}}(X,\mathbb{Z}(p))'' \to H^{2p-1}_{\mathcal{D}}(X\setminus\Sigma,\mathbb{Z}(p))'' \tag{4.3.2}$$

is injective by the localization sequence. So we may replace X with $X\setminus\Sigma$, and assume that Z is smooth, and hence irreducible. Here we may assume also that the closure $\overline{Z}$ of Z in $\overline{X}$ is a good smooth compactification (i.e. $\overline{Z}\setminus Z$ is a divisor with normal crossings) by taking further blowing-ups if necessary, and that every irreducible component of $\overline{Z}\setminus Z$ is contained by only one irreducible component of $\overline{X}\setminus X$. Then the isomorphism (4.1.1) is compatible with the push-forward by the closed embedding $Z\to X$, and the assertion is reduced to the case $X=Z$, $p=1$.

By (1.3) we have isomorphisms

$$H^1_{\mathcal{D}}(X,\mathbb{Z})'' = H^1_{\mathcal{D}}(X,\mathbb{Z}) = H^1(\overline{X}, C^\bullet_{\overline{X}\langle D\rangle}\mathbb{Z}(1)),$$

and the cycle map is calculated by using the commutative diagram

$$\begin{array}{ccc} \Gamma(X,\mathbb{G}_m) & \xrightarrow{\sim} & H^1(\overline{X}, C^\bullet_{\overline{X}\langle D\rangle}\mathbb{Z}(1)) \\ \downarrow & & \downarrow \\ \Gamma(X^{\mathrm{an}},\mathcal{O}^*_{X^{\mathrm{an}}}) & \xrightarrow{\sim} & H^1(X, C^\bullet_X\mathbb{Z}(1)), \end{array} \tag{4.3.3}$$

where the isomorphism on the bottom row is induced by the canonical quasi-isomorphism

$$\mathcal{O}^*_{X^{\mathrm{an}}} = C(\mathbb{Z}_{X^{\mathrm{an}}}(1)\to\mathcal{O}_{X^{\mathrm{an}}}).$$

Here we may replace $H^1(X, C_X^\bullet\mathbb{Z}(1))$ with the cohomology of the single complex associated with

$$S^\bullet(X)(-d) \to \mathcal{C}^\bullet(X)/F^{-d}\mathcal{C}^\bullet(X).$$

Then, using (1.2.2), it is enough to show that the image of $g \in \Gamma(X^{\mathrm{an}}, \mathcal{O}^*_{X^{\mathrm{an}}})$ in this cohomology is represented by

$$(-(2\pi i)^{-d}\gamma, \quad (2\pi i)^{-d-1}\log_{\mathrm{Hv}} g),$$

because the vertical morphisms of (4.3.3) are injective.

We can verify this assertion by using a Cech resolution together with the delta functions supported on faces of a triangulation of X^{an} compatible with γ. (See [28] for the notion of integral current.) Indeed, let $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ be an open covering of X^{an} such that U_i are simply connected. We will denote by $C_{\mathcal{U}}^\bullet\mathcal{F}$ the Cech complex of a sheaf $\mathcal{F}$ associated with the covering $\mathcal{U}$, where

$$C_{\mathcal{U}}^i\mathcal{F} = \textstyle\bigoplus_{|I|=i+1}\Gamma(U_I, \mathcal{F}) \quad \text{with } U_I = \cap_{i\in I}U_i \text{ for } I \subset \Lambda.$$

For $g \in \Gamma(X^{\mathrm{an}}, O^*_{X^{\mathrm{an}}})$, we have the corresponding element in the cohomology of the single complex associated with

$$C_{\mathcal{U}}^\bullet\mathbb{Z}_{X^{\mathrm{an}}}(1) \to C_{\mathcal{U}}^\bullet\mathcal{O}_{X^{\mathrm{an}}}$$

(where the first term has degree one), and it is given by

$$(\{(\log(g|_{U_i}) - \log(g|_{U_j}))|_{U_{i,j}}\}_{i>j}, \{\log(g|_{U_i})\}_i) \in C_{\mathcal{U}}^1\mathbb{Z}_{X^{\mathrm{an}}}(1) \oplus C_{\mathcal{U}}^0\mathcal{O}_{X^{\mathrm{an}}}.$$

We define $C_{\mathcal{U}}^i\mathcal{C}^j(X)$, $C_{\mathcal{U}}^iS^j(X)$ similarly. Then

$$\{(2\pi i)^{-d-1}((\log_{\mathrm{Hv}} g)|_{U_i}) - \log(g|_{U_i}))\}_i \in C_{\mathcal{U}}^0\mathcal{C}^{-2d-2}(X)$$

belongs to the image of $C_{\mathcal{U}}^0S^{-2d-2}(X)(-d)$. So we get the assertion, using the triple complex

$$C_{\mathcal{U}}^\bullet S^\bullet(X)(-d) \to C_{\mathcal{U}}^\bullet(\mathcal{C}^\bullet(X)/F^{-d}\mathcal{C}^\bullet(X)).$$

This completes the proof of (4.3).

4.4. Remark. The cycle map (2.4.1) induces

$$\text{(4.4.1)} \qquad \mathrm{CH}^p(X,1) \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, H^{2p-1}(X,\mathbb{Z})(p)),$$

and (4.3) implies that the image of ζ by this morphism is represented by

$$-(2\pi i)^{-d}\gamma \quad \text{and} \quad -(2\pi i)^{-d-1}\textstyle\sum_j(i_j)_*(g_j^{-1}dg_j).$$

Let $\mathrm{CH}^p_{\mathrm{hom}}(X,1)$ be the kernel of (4.4.1). Then, in the notation of (1.1), the cycle map (2.4.1) induces the higher Abel-Jacobi map

$$\text{(4.4.2)} \qquad \mathrm{CH}^p_{\mathrm{hom}}(X,1) \to J(H^{2p-2}(X,\mathbb{Z})(p)) = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^{2p-2}(X,\mathbb{Z})(p)).$$

Assume $H^{2p-1}(X,\mathbb{Q})=0$ or X proper. Then $\mathrm{CH}^p(X,1)/\mathrm{CH}^p_{\mathrm{hom}}(X,1)$ is finite because the target of (4.4.1) is torsion. So (4.4.2) induces the higher Abel-Jacobi map

$$\mathrm{CH}^p(X,1)_{\mathbb{Q}} \to J(H^{2p-2}(X,\mathbb{Q})(p)). \tag{4.4.3}$$

By (4.3) this is expressed explicitly as follows. For $\zeta\in\mathrm{CH}^p_{\mathrm{hom}}(X,1)$, there exist a C^∞ chain Γ on X and $\Xi\in F^{-d}\mathcal{C}^{-2d-1}(\overline{X}\langle D\rangle)$ such that

$$\partial\Gamma=\gamma,\quad d\Xi=(g^{-1}dg)^\wedge.$$

By (4.3) and (1.2.2), the image of ζ under the higher Abel-Jacobi map (4.4.3) is represented by the current

$$\Phi_\zeta=(2\pi i)^{-d-1}(\textstyle\sum_j (i_j)_*\log_{\mathrm{Hv}} g_j+(2\pi i)\iota\Gamma-\Xi|_X). \tag{4.4.4}$$

(Note that $d\iota\Gamma=-\iota\gamma$ by Stokes.) If X is proper, it is enough to consider $\Phi_\zeta(\omega)$ for C^∞ forms ω which are direct sums of forms of type (i,j) with $i\ge d+1$, because the dual of $H^{2p-2}(X,\mathbb{C})/F^pH^{2p-2}(X,\mathbb{C})$ is $F^{d+1}H^{2d+2}(X,\mathbb{C})$. Then Ξ can be neglected, and we get the higher Abel-Jacobi map in [5], [37] (see also [32]).

4.5. Remark. With the notation of (1.1) and (2.2), the image of $\mathrm{CH}^{p-1}_{\mathrm{dec}}(X,1)_{\mathbb{Q}}$ by the higher Abel-Jacobi map (4.4.3) is contained in

$$J(N^{p-1}H^{2p-2}(X,\mathbb{Q})(p))\subset \mathrm{Hdg}^{p-1}(X)_{\mathbb{Q}}\otimes_{\mathbb{Z}}\mathbb{C}^*,$$

where $N^{p-1}H^{2p-2}(X,\mathbb{Q})$ is the $\mathbb{Q}$-submodule generated by algebraic cycle classes, and $\mathrm{Hdg}^{p-1}(X)_{\mathbb{Q}}$ is the group of Hodge cycles with rational coefficients. This can be reduced to the case $p=1$ by using resolutions of singularities, see e.g. [40]. The induced map

$$\mathrm{CH}^p_{\mathrm{ind}}(X,1)_{\mathbb{Q}}\to J(H^{2p-2}(X,\mathbb{Q})(p))/\mathrm{Hdg}^{p-1}(X)_{\mathbb{Q}}\otimes_{\mathbb{Z}}\mathbb{C}^* \tag{4.5.1}$$

is called the reduced Abel-Jacobi map.

By [43], [46], the kernel of (4.5.1) for $p=2$ is isomorphic to the cokernel of

$$K_2(\mathbb{C}(X))_{\mathbb{Q}}\to \varinjlim_U \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q},H^2(U,\mathbb{Q})(2)) \tag{4.5.2}$$

where the morphism is given by $d\log\wedge d\log$ at the level of integral logarithmic forms, and the inductive limit is taken over nonempty open subvarieties U of X.

Indeed, for a divisor Z on X, we have $H^3_Z(X,\mathbb{Q})=H^{\mathrm{BM}}_{2n-3}(Z,\mathbb{Q})(-n)$ and

$$\mathrm{CH}^1_{\mathrm{ind}}(Z,1)_{\mathbb{Q}}=\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q},H^3_Z(X,\mathbb{Q})(2))$$

by (0.3) and (1.1.2). Therefore, if Z is sufficiently large, we get a short exact sequence

$$\begin{aligned}0\to H^2(X,\mathbb{Q})/N^1H^2(X,\mathbb{Q})&\to H^2(X\setminus Z,\mathbb{Q})\\ &\to \mathrm{Ker}(H^3_Z(X,\mathbb{Q})\to H^3(X,\mathbb{Q}))\to 0,\end{aligned}$$

and from its associated long exact sequence we can deduce

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^2(X\setminus Z,\mathbb{Q})(2))\\ &= \mathrm{Ker}(\mathrm{CH}^1_{\mathrm{ind}}(Z,1)_{\mathbb{Q}} \to J((H^2(X,\mathbb{Q})/N^1H^2(X,\mathbb{Q}))(2))),\end{aligned}$$

because $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^j(X,\mathbb{Q})(2)) = 0$ for $j = 2, 3$. Then it is enough to take the inductive limit over Z, and divide it by the image of $K_2(\mathbb{C}(X))_{\mathbb{Q}}$ under the tame symbol.

Note that the morphism $H^2(X\setminus Z,\mathbb{Q}) \to H^3_Z(X,\mathbb{Q})$ is given by taking the residue of logarithmic forms (at least on Z_{reg}, see [22]), and the residue of $d\log f \wedge d\log g$ coincides with the differential of the logarithm of the image of $\{f, g\}$ by the tame symbol up to sign. The image of (4.5.1) is countable for $p \geq 2$ by the rigidity argument of A. Beilinson [4] and S. Müller-Stach [40], and (4.5.1) is not necessarily injective for $p \geq 3$ (see [20]). For $p = 2$, it is conjectured that $\mathrm{CH}^2_{\mathrm{ind}}(X,1)_{\mathbb{Q}}$ should be countable by C. Voisin [55], and that (4.5.2) should be surjective by A. Beilinson [5].

It does not seem easy to prove the last conjecture by using [39]. Indeed, let I be the target of (4.5.2) with $\mathbb{Q}$ replaced by $\mathbb{Z}$, and I' be the image of $I'' := K_2(\mathbb{C}(X))$ in I. Then $I''/m = I'/m = I/m$ for any positive integer m by loc. cit. (using the exact sequence (3.7.1)). Hence I/I' is divisible. It is torsion-free by the snake lemma, because so is I. Therefore, I/I' is uniquely divisible as proved in [46] and we cannot get any more information.

5. Construction of indecomposable higher cycles

5.1. Elliptic surfaces. Let $\pi : X \to C$ be an elliptic surface over a smooth proper curve (i.e., X is smooth, π is proper, and general fibers of π are elliptic curves). Let Σ denote the smallest subset of C such that π is smooth over $C' := C \setminus \Sigma$. Put $X' = \pi^{-1}(C')$, $X_c = \pi^{-1}(c)$ for $c \in C$. It is well-known that the higher direct image sheaf $R^1\pi_*\mathbb{Q}_X$ is an intersection complex (up to a shift) by the decomposition theorem [7], and $H^1(C, R^1\pi_*\mathbb{Q}_X)$ is a pure Hodge structure of weight 2, see [57]. Let L be the Leray filtration on $H^2(X,\mathbb{Q})$ which is defined by

$$\begin{aligned}&L^1H^2(X,\mathbb{Q}) = \cap_{c\in C}\mathrm{Ker}(H^2(X,\mathbb{Q}) \to H^2(X_c,\mathbb{Q})),\\ &L^2H^2(X,\mathbb{Q}) = \pi^*H^2(C,\mathbb{Q}),\end{aligned}$$

and $L^0H^2(X,\mathbb{Q}) = H^2(X,\mathbb{Q}), L^3H^2(X,\mathbb{Q}) = 0$. Then

$$\mathrm{Gr}^j_L H^2(X,\mathbb{Q}) = H^j(C, R^{2-j}\pi_*\mathbb{Q}_X).$$

We say that an elliptic surface has *essentially no nontrivial section* if

$$\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^1(C, R^1\pi_*\mathbb{Q}_X)(1)) = 0. \tag{5.1.1}$$

By [57] this condition is equivalent to that the sections of $\pi : X \to C$ are torsion, if the monodromy of the local system $R^1\pi_*\mathbb{Q}_X|_{C'}$ is nontrivial and X is given a 0-section.

Let $\{\pi_t : X_t \to C_t\}_{t\in\Delta}$ be a family of elliptic surfaces over an open disk Δ of radius ε whose restriction over Δ^* $(:= \Delta \setminus \{0\})$ is locally topologically trivial. We may view $\{\pi_t : X_t \to C_t\}_{t\in\Delta}$ as a small deformation of $\pi_0 : X_0 \to C_0$ or a degeneration of elliptic surfaces. We say that it is *cohomologically nondegenerate* if $\{\mathrm{Gr}_L^1 H^2(X_t, \mathbb{Q})\}_{t\in\Delta}$ is a constant local system on Δ. Note that $\dim \mathrm{Gr}_L^1 H^2(X_t, \mathbb{Q})$ is constant if and only if so is $\dim \mathrm{Gr}_L^0 H^2(X_t, \mathbb{Q})$. Using the decomposition theorem [7], the latter dimension is given by $\sum_{c\in C_t}(n_{t,c} - 1) + 1$, where $n_{t,c}$ is the number of irreducible components of $X_{t,c}$.

Let I denote the open interval $(0, \varepsilon)$ contained in $\Delta^* := \Delta \setminus \{0\}$. Let $\{\zeta_t\}_{t\in I}$ be an analytic family of higher cycles with $\zeta_t \in \mathrm{CH}^2(X_t, 1)$. We say that it *degenerates topologically* at $t = 0$, if there are C^∞ families $\{\gamma_t\}_{t\in I}, \{\Gamma_t\}_{t\in I}$ as in (4.4) which degenerate to a point as $t \to 0$.

5.2. Theorem. *Let $\{\pi_t : X_t \to C_t\}_{t\in\Delta}$ and $\{\zeta_t\}_{t\in I}$ be as above. Assume that $\{X_t\}$ is cohomologically nondegenerate, X_0 has essentially no nontrivial section, $\{\zeta_t\}$ degenerates topologically at $t = 0$, and $\int_{\Gamma_t} \omega_t \neq 0$ for some $\omega_t \in \Gamma(X_t, \Omega^2_{X_t})$ where $t \in I$ is generic and Γ_t is as above. Then for a general $t \in I$, the transcendental part of the image of ζ_t by the reduced Abel-Jacobi map* (4.5.1) *does not vanish* (*i.e. its image in the Jacobian is not contained in the image of $F^1H^2(X_t, \mathbb{C})$*)*, and hence $\zeta_t \neq 0$ in $\mathrm{CH}^2_{\mathrm{ind}}(X_t, 1)_{\mathbb{Q}}$.*

Proof. Let $\mathcal{V}_t = \mathrm{Gr}_F^0 \mathrm{Gr}_L^1 H^2(X_t, \mathbb{Q})$. Then $\{\mathcal{V}_t\}_{t\in\Delta}$ is a holomorphic vector bundle on Δ. By integrating holomorphic 2-forms on Γ_t, we get an analytic section σ_Γ of $\{\mathcal{V}_t\}_{t\in I}$, because $H^2(C_t, \mathbb{Q})$ and the image of $H^2(X_t, \mathbb{Q}) \to H^2(X_{t,c}, \mathbb{Q})$ are of type $(1, 1)$. Let L denote also the dual filtration on $H_2(X_t, \mathbb{Q})$ such that $\mathrm{Gr}^L_j H_2(X_t, \mathbb{Q})$ is the dual of $\mathrm{Gr}_L^j H^2(X_t, \mathbb{Q})$, i.e.

$$L_0 H_2(X_t, \mathbb{Q}) = \mathrm{Im}(\textstyle\bigoplus_c H_2(X_{t,c}, \mathbb{Q}) \to H_2(X_t, \mathbb{Q})),$$
$$L_1 H_2(X_t, \mathbb{Q}) = \mathrm{Ker}(\pi_* : H_2(X_t, \mathbb{Q}) \to H_2(C_t, \mathbb{Q})).$$

Let $\{\eta_t\} \in \{\mathrm{Gr}^L_1 H_2(X_t, \mathbb{Q})(-1)\}_{t\in\Delta}$ be a continuous family of topological cycles with rational coefficients. Then it determines a holomorphic section σ_η of $\{\mathcal{V}_t\}_{t\in\Delta}$ by integrating forms on it. By (4.4) the image of ζ_t by (4.4.3) in $J(H^2(X_t, \mathbb{Q})(2))$ modulo the image of $F^1H^2(X_t, \mathbb{C})$ is given by σ_Γ modulo the image of $H_2(X_t, \mathbb{Q})(-1)$. If it vanishes for any $t \in I$, then there exists a family $\{\eta_t\}_{t\in\Delta}$ as above such that σ_Γ coincides with σ_η over I, because I is an uncountable set. By hypothesis, the limit of σ_Γ for $t \to 0$ is zero, and so is the value of σ_η at the origin. This implies that η_0 belongs to

$$\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, \mathrm{Gr}_L^1 H^2(X_0, \mathbb{Q})(1))$$

using Poincaré duality. Therefore $\eta_0 = 0$ by (5.1.1), and hence $\eta_t = 0$ for any $t \in \Delta$ by the triviality of the local system. Thus we get $\sigma_\Gamma = 0$. But this contradicts the nonvanishing of $\int_{\Gamma_t} \omega_t$.

5.3. Construction. Let $Y'_t = \mathbb{C}^2, S' = \mathbb{C}$, and $g_t : Y'_t \to S'$ be a polynomial map defined by $g_t = y^2 - x^2(x + t)$ for $t \in \mathbb{C}$. Taking the closure of the graph in $\mathbb{P}^2 \times S'$, we get $\overline{g}_t : \overline{Y}'_t \to S'$, and this gives an elliptic surface $f_t : Y_t \to S := \mathbb{P}^1$ by taking the minimal model of the singular fiber over $\infty \in \mathbb{P}^1$ using Kodaira's classification.

If $t \neq 0$, the singular fibers over 0 and $-4t^3/27$ are both rational curves with one ordinary double point. (For example, if we put $s = y/x$, then $x = s^2 - t, y = s(s^2 - t)$ on $\{g_t(x, y) = 0\}$.) So each of these singular fibers determines a higher cycle by taking the normalization and choosing a rational function with simple zero and pole at the pull-back of the double point. This is well-defined up to a sign and modulo decomposable cycles. It t is positive, we can choose the rational function on the singular fiber at 0 so that γ and Γ in (4.2) and (4.4) are given respectively by

$$\gamma'_t = \{(x, y) \in \mathbb{R}^2 : g_t(x, y) = 0, x \leq 0\},$$
$$\Gamma'_t = \{(x, y) \in \mathbb{R}^2 : g_t(x, y) \leq 0, x \leq 0\},$$

where the function is $-(y - \sqrt{t}x)/(y + \sqrt{t}x) = -(s - \sqrt{t})/(s + \sqrt{t})$.

The higher cycles constructed above are still decomposable because Y_t is a rational surface, and we have to take a base change. Let $\rho : C \to S$ be a generic hyperelliptic curve such that $\infty \in S\,(= \mathbb{P}^1)$ is a ramification point, but $0 \in S$ is not. We assume that $H^1(C, \mathbb{Q})$ does not contain a Hodge structure isomorphic to the cohomology of the elliptic curve defined by $y^2 = x^3 + 1$ (with j-invariant 0). Let $\widetilde{f}_t : X_t \to C$ be the minimal model of the base change of $f_t : Y_t \to S$ by ρ. (Actually, we can also consider the open surface with the singular fiber over ∞ deleted, because 1 is not an eigenvalue of the local monodromy of $R^1(f_t)_*\mathbb{Q}_{Y_t}|_{S'}$ around ∞ so that $R^1(f_t)_*\mathbb{Q}_{Y_t} = \mathbf{R}j_*j^{-1}R^1(f_t)_*\mathbb{Q}_{Y_t}$ where $j : S' \to S$ denotes the inclusion, and similarly for the pull-back by ρ.) We assume that ρ is not ramified over 0, and choose a point $\widetilde{0}$ of C over 0. Let γ_t be the connected component of the pull-back of γ'_t contained in the fiber over $\widetilde{0}$. There is a connected component Γ_t of the pull-back of Γ'_t such that $\partial\Gamma_t = \gamma_t$ for t sufficiently small.

5.4 Remarks. (i) The image of Γ'_t by g_t is the interval $[-4t^3/27, 0]$, and Γ'_t gives a degeneration of $\gamma'_t \subset Y_{t,0}$ as $c \to -4t^3/27$, i.e. γ'_t is the vanishing cycle associated to the singular fiber over $-4t^3/27$. The cohomological nondegeneration is related to the phenomenon that two A_1-singularities appear by a deformation of a holomorphic function with an isolated singularity of type A_2, i.e. two A_1-singularities of a function can join and degenerate to an A_2-singularity.

(ii) Instead of the hyperelliptic curve C, it is also possible to consider an m-fold cyclic covering of $\mathbb{P}^1$ ramified over two points α and ∞ if α is generic and m is prime to 6 and strictly greater than 6.

5.5. Theorem. *If ε is sufficiently small, then $\{X_t\}$ satisfies the assumptions of* (5.2).

Indeed, the hypotheses are satisfied by the following lemmas and proposition:

5.6. Lemma. *The family $\{X_t\}$ is cohomologically nondegenerate.*

Proof. For $t \in \Delta$ we have

$$H^1(S', R^1(f_t)_*\mathbb{Q}_{Y_t}|_{S'}) = H^1(S', R^1(g_t)_*\mathbb{Q}_{Y'_t}) = 0,$$

see e.g. [24]. In particular, these groups are constant for $t \in \Delta$, and this holds also for the cohomology of its restriction over a small open disk with center 0 in S' if ε is sufficiently

small, because there are no singular fibers over the complement of the disk in S' if t is sufficiently small. (Note that by hypothesis, the direct image sheaf is defined over $\cup S'_t$ and its restriction to each S'_t is $R^1(g_t)_*\mathbb{Q}_{Y'_t}$; moreover, a similar assertion holds for the pull-back by ρ.) So we can use the Mayer-Vietoris sequence to show that the cohomology of the pull-back of the sheaf to C is constant, and the assertion follows.

5.7. Lemma. *The elliptic surface X_0 has essentially no nontrivial section.*

Proof. The local system associated to f_0 has a finite monodromy group, and is trivialized by taking the pull-back under a finite base change. This holds also for the local system associated to X_0, and it is enough to show that there is essentially no nontrivial section (see (5.1.1)) for the constant elliptic surface over C whose fiber X_c is defined by the equation $y^2 = x^3 + 1$. But the self-duality of $H^1(X_c, \mathbb{Q})$ implies

$$\mathrm{Hom}(\mathbb{Q}, H^1(X_c, \mathbb{Q}) \otimes H^1(C, \mathbb{Q})(1)) = \mathrm{Hom}(H^1(X_c, \mathbb{Q}), H^1(C, \mathbb{Q})),$$

and the assertion follows from the hypothesis.

5.8. Proposition. *The integral $\int_{\Gamma_t} \omega_t$ does not vanish for some holomorphic 2-from ω_t.*

Proof. We may assume that C is given by the equation $s^2 = h(z)$ with

$$h(z) = \prod_{j=1}^{2g+1} (z - \alpha_j),$$

where z is the coordinate of S', the α_j are generic complex numbers, and $g \geq 1$. Then $dz/\sqrt{h(z)}$ is a nonzero 1-form on C. We show that there is a nonzero 2-form ω_t on X_t whose restriction to $Y'_t \times_{S'} C'$ is $dx \wedge dy/f_t^*\sqrt{h(z)}$, where C' is the inverse image of S'. (This is rather trivial if we assume that the genus of C is sufficiently large, compared with the order of the pole of $dx \wedge dy/f_t^*dz$.)

Let $\omega_{\mathrm{rel}} = dx \wedge dy/f_t^*dz$. It gives a section of $(f_t)_*\omega_{Y_t/S}$ by [36], because ω_{rel} satisfies the Gauss hypergeometric differential equation

$$z(z + 4t^3/27)\partial_z^2\omega_{\mathrm{rel}} + (2z + 4t^3/27)\partial_z\omega_{\mathrm{rel}} + (5/36)\omega_{\mathrm{rel}} = 0,$$

(via the Gauss-Manin connection, see e.g. [24]), and the roots of its indicial equation [18] at $0, -4t^3/27, \infty$ are respectively $\{0, 0\}, \{0, 0\}$ and $\{1/6, 5/6\}$. Similarly, $\rho^*\omega_{\mathrm{rel}}$ is extended to a section of (the direct image of) $\omega_{X_t/C}$, because the roots of the indicial equation at $\rho^{-1}(\infty)$ are $1/3, 5/3$. Then it gives a nonzero section of ω_{X_t} by using the section $dz/\sqrt{h(z)}$ of ω_C, and it coincides with the above 2-form ω_t.

To show $\int_{\Gamma_t} \omega_t \neq 0$, we may assume that the α_j are real and sufficiently small so that $f_t^*\sqrt{h(z)} > 0$ on Γ_t. Then the assertion is clear.

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